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ALSO,

SERVING AS AN INTRODUCTION TO THE

"TREATISE ON ALGEBRA."

REVISED EDITION.

BY GEORGE R. PERKINS, LL. D.

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PREFACE

TO THE SECOND EDITION.

OUR Common Schools have reached that stage of advancement which enables them to pursue a more extended course of studies than formerly.

A few years ago Geographical Studies were hardly known in our Common Schools; now, every child is taught, at least, the elements of Geography. In the same way Algebra is fast finding its proper place among other branches of an elementary education.

In a logical point of view, there is perhaps no science so well calculated to awaken a vigorous and rigid exercise of the reasoning powers as Mathematics.

Algebra and Geometry are the two great pillars of this science. Algebra, being more nearly allied to Arithmetic, may be made to precede the study of Geometry. Indeed, Algebra is a sort of universal Arithmetic, and affords great assistance to a clear and correct comprehension of the various Arithmetical rules in frequent use.

It cannot, however, be expected, at present, that our Common Schools should pursue Algebra to the same extent as pursued in our Colleges; still they may, to good advantage, acquire the more elementary portions of this branch of Mathematics.

With these views before me, I have endeavored to prepare a clear and concise exposition of the Elements of this important branch of Mathematics, which should be adapted to the present wants of our Common Schools.

The plan of this work is quite similar to that of my "Treatise on Algebra," and may in some respects be regarded as an Introduction to that work.

Under Chapter VII., I have introduced a method of elimination by *Indeterminate Multipliers*. This method was first given, I believe, by the celebrated LAGRANGE, in his *Mécanique Analytique*. I have given also, under this chapter, an entirely *new method* of solving three simultaneous simple equations, which, from the peculiar process employed, may be called *The Chequer-Board Method*.

Under Chapter VIII., I have also thought it not out of place to give a few examples immediately after Permutations and Combinations, on the Theory of Probabilities.

Should these few Elements of Algebra be found to aid in elevating the standard of our Common-School education, I shall feel that my object in preparing this book has been accomplished.

GEO. R. PERKINS.

UTICA, March, 1846.

PREFACE

TO REVISED EDITION.

In the present edition we have made such additions, explanations, and modifications, as a long use of this book in actual class exercises, seemed to point out as necessary.

During a period of eight years the author has used this work in the New York State Normal School. The large number of pupils thus coming under his instruction, possessing, in a high degree, a spirit of inquisitiveness, as well as of close investigation, have afforded him a rare opportunity of observing the defects, as well as faults of his book.

He has endeavored to supply these defects, and to correct these faults; and at the same time to add much that will be interesting, as well as useful, to the mathematical student.

GEO. R. PERKINS.

UTICA, March, 1854.

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ELEMENTS OF ALGEBRA.

CHAPTER I.

DEFINITIONS AND PRELIMINARY RULES.

DEFINITIONS AND SYMBOLS.

(*Article* 1.) **Algebra** is that branch of Mathematics, in which the operations are performed by means of *figures*, *letters*, and *signs* or *symbols*.

(2.) In Algebra, quantities, whether given or required, are usually represented by letters. The first letters of the alphabet are, for the most part, used to represent known quantities; and the final letters are used for the unknown quantities.

(3.) The symbol $=$, is called the sign of *equality;* and denotes that the quantities between which it is placed, are equal or equivalent to each other. Thus, \$1=100 *cents*, which is read, one dollar equals one hundred cents. In the same way $a=b$, is read, a equals b; and the same for other similar expressions.

(4.) The symbol $+$, is called the sign of *addition*, or *plus;* and denotes that quantities between which it is placed, are to be added together. Thus $a+b=c$, is read, a and b added, equals c. Again, $a+b+c=d+x$, is read, a, b, and c added, equals d and x added.

(5.) The symbol $-$, is called the sign of *subtraction*, or *minus;*

and denotes that the quantity which is placed at the right of it is to be subtracted from the quantity on the left. Thus, $a-b=c$, is read, a diminished by b equals c.

(6.) When algebraic quantities are written without any sign prefixed, they are understood to be affected by the sign *plus*, and the quantities are said to be *positive* or *affirmative;* but quantities having the sign *minus* prefixed, are called *negative* quantities. Thus, a is the same as $+a$, b is the same as $+b$, and each is called a positive quantity; while $-a$, $-b$, are called negative quantities.

(7.) The symbol $\times$, is called the sign of *multiplication;* and denotes that the quantities between which it is placed, are to be multiplied together. Thus, $a\times b=c$, is read, a multiplied by b equals c. Multiplication is also represented by placing a *point* or *dot* between the factors, or terms to be multiplied. Thus, $a.b$ is the same as $a\times b$. Another method frequently employed, is to unite the quantities in the form of a word. Thus, abc is the same as $a.b.c$, or $a\times b\times c$.

(8.) The symbol $\div$, is called the sign of *division;* and denotes that the quantity on the left of it is to be divided by the quantity on the right. Thus, $a\div b=c$, is read, a divided by b equals c. Division is also indicated by placing the divisor under the dividend, with a horizontal line between them, like a vulgar fraction. Thus, $\frac{x}{y}$ is the same $x\div y$.

(9.) When quantities are inclosed in a parenthesis, brace, or bracket, they are to be treated as a simple quantity. Thus, $(a+b)\div c$, indicates that the sum of a and b is to be divided by c. Again, each of the expressions, $(x-y)\div z$; $\{x-y\}\div z$; $[x-y]\div z$, is read, y subtracted from x, and the remainder divided by z. The same thing may be expressed by drawing a horizontal line, or bar, over the compound quantity, which line or bar is called a *vinculum*. Thus, $\overline{x+y}\times z$, denotes, that the sum of x and y is to be multiplied by z.

(10.) When a quantity is added to itself several times, as $c+c+c+c$, we may write it but once, and prefix a number to show how many times its value has been repeated by this addition. Thus, $c+c+c$ is the same as $3c$; $d+d+d+d$ is the same as $4d$. The numbers thus prefixed to the quantities are called *coefficients*. Thus, in the expressions $3c$, $4d$, the coefficients of c and d are 3 and 4 respectively. A coefficient may consist itself of a letter. Thus, in the expression nx, n may be regarded as the coefficient of x; so also may x be considered as the coefficient of n.

When no coefficient is written, the quantity is regarded as having 1 for its coefficient. Thus, a is the same as $1a$, and xy is the same as $1xy$.

(11.) The continued product of a quantity into itself is usually denoted by writing the quantity once, and placing a number over the quantity, a little to the right. Thus $a\times a\times a$ is the same as a^3. The number thus placed over the quantity, is called the *exponent* of the quantity, and denotes the number of equal factors which are to be multiplied together. Also, $a\times a\times a\times a$ is the same as a^4.

When a quantity is written without any exponent, it is understood that its exponent is 1. Thus, a is the same as a^1, and $(x+y)\times m$ is the same as $(x+y)^1\times m^1$.

(12.) The *reciprocal* of a quantity is the value of a unit when divided by that quantity. Thus, $\frac{1}{a}$ is the reciprocal of a; also $\frac{1}{b}$ is the reciprocal of b.

(13.) An *Algebraic expression* is any combination of letters and numbers, formed by the aid of algebraic signs, in conformity with the foregoing definitions.

An algebraic expression composed of two or more terms connected by + or —, is called a *polynomial*. A polynomial composed of but two terms, is called a *binomial;* one composed of three terms, is called a *trinomial*.

Thus,

$$\left.\begin{array}{l} 3a+5b \\ 7x^2-3y^3 \\ 3a-x^4 \end{array}\right\} \text{are binomials.}$$

$$\left.\begin{array}{l} 3a^2+2b-x \\ 4m+y+a \\ 5h-x+d \end{array}\right\} \text{are trinomials.}$$

ADDITION.

(14.) Addition, in Algebra, is finding the simplest expression for several algebraic quantities, connected by + or —.

Suppose we wish to find the sum of 3 apples, 7 apples, 4 apples, and 10 apples. From what we know of arithmetic, we should proceed as in the following

OPERATION.

3 apples,
7 apples,
4 apples,
10 apples,
―――――
24 apples = sum of all.

If, instead of writing the word *apples*, we write only *a*, its initial letter, we shall have this second

OPERATION.

$$\begin{array}{r} 3a \\ 7a \\ 4a \\ 10a \\ \hline 24a \end{array} = \text{sum of all the } a\text{'s.}$$

If a, instead of representing an apple, stands for any other thing, then would the sum of $3a$, $7a$, $4a$, and $10a$, be accurately represented by $24a$.

Frequently, in algebra, the quantities to be united or added, are all placed in the same horizontal line. The above quantities, when placed after this method, become:

$$3a+7a+4a+10a=24a.$$

(15.) We may remark that in algebra the results are, in general, true, whatever values are given to the letters. In this example, suppose $a=5$, and we shall have this

OPERATION.

$$\begin{array}{rcrcr} 3a & = & 3\times5 & = & 15 \\ 7a & = & 7\times5 & = & 35 \\ 4a & = & 4\times5 & = & 20 \\ 10a & = & 10\times5 & = & 50 \\ \hline 24a & = & 24\times5 & = & 120 \end{array}$$

Again, suppose $a=2$, and we have this second

OPERATION.

$$\begin{array}{rcrcr} 3a & = & 3\times2 & = & 6 \\ 7a & = & 7\times2 & = & 14 \\ 4a & = & 4\times2 & = & 8 \\ 10a & = & 10\times2 & = & 20 \\ \hline 24a & = & 24\times2 & = & 48 \end{array}$$

As a second example, suppose James to play 6 successive games at marbles. The first game he wins 6 marbles; the second game he loses 3 marbles; the third game he wins 10 marbles; the fourth game he loses 8 marbles; the fifth game he wins 4 marbles; the sixth game he loses 5 marbles. How many marbles will he now have, on the supposition that he had none before playing?

It is obvious that if we subtract the sum of his *losses* from the sum of his *gains*, we shall obtain the number which he finally has.

If we consider the gains as *positive* quantities, we ought to consider his losses as *negative* quantities.

The sum of his gains is found by this

OPERATION.

$$\begin{array}{r} 6m \\ 10m \\ 4m \\ \hline 20m \end{array} = \text{sum of gains.}$$

The sum of his losses is found by the following

OPERATION.

$$\begin{array}{r} -3m \\ -8m \\ -5m \\ \hline -16m \end{array} = \text{sum of losses.}$$

Hence, $20m-16m=4m$=his final number of marbles; that is, he had 4 marbles after playing the sixth game.

If m had represented \$100, instead of one marble, then at the end of the sixth game, James would have had $4m=4\times\$100=\$400$.

(16.) From this we see that negative quantities are added in the same way as positive quantities, observing to prefix the — sign to the sum.

If we consider distances measured in a certain direction as positive, then will distances measured in an opposite direction be negative. Thus, calling north latitude positive, south latitude must be considered as negative.

Suppose a ship, in 20 degrees north latitude, to sail 15 degrees south, then 6 degrees north, then 30 degrees south, then 5 degrees north, and finally to go 10 degrees south. What latitude is she in?

Counting from the equator she is already 20 degrees north, to which adding the northings made, we have

$$20d+6d+5d=31d,$$

for the latitude in which she would have been, provided she had made no southing. Where it may be remarked that we use *d* to represent one degree.

The sum of the southings is

$$-15d-30d-10d=-55d.$$

Uniting the sum of the northings with the sum of the southings, we get

$$31d-55d=-24d.$$

Now since the result is —, it shows that she is south of the equator, that is, she is in 24 degrees south latitude.

(17.) It must always be borne in mind that the office of the negative sign is to denote a state, condition, or effect, directly opposite to that denoted by the positive sign.

If the credits in a book account are considered as positive, the debts must be made negative.

If the degrees of a thermometer above zero are called +, then those below must be called —.

We are now prepared to give a rule corresponding to

CASE I.

(18.) *When the quantities are alike but have unlike signs.*

RULE.

I. Find the sum of the coefficients of the positive quantities; and also the sum of the coefficients of the negative quantities.

II. Subtract the LESS *sum from the* GREATER.

III. Prefix the sign of the greater sum to the remainder, and annex the common letter or letters.

1. Find the sum of

$$3c-2c+5c-4c.$$

The sum of the coefficients of the positive terms is

$$+3+5=8.$$

The sum of the coefficients of the negative terms is

$$-2-4=-6.$$

The difference is 2, to which, giving the sign of the greater, annexing the common letter, we have $+2c$, or simply $2c$, for the sum sought. (See Art. 6.)

In practice, we usually write all the different terms under each other, as in Arithmetic. Thus,

$$\begin{array}{r l}
3c & \\
-2c & \\
5c & \\
-4c & \\
\hline
2c & =\text{sum sought.}
\end{array}$$

Let this same example be wrought on the supposition that $c=10$.

$$\begin{array}{r r r}
3c= & 3\times10= & 30 \\
-2c= & -2\times10= & -20 \\
5c= & 5\times10= & 50 \\
-4c= & -4\times10= & -40 \\
\hline
2c= & 2\times10= & 20
\end{array}$$

When $c=\frac{1}{2}$ it becomes

$$\begin{array}{r r r}
3c= & 3\times\frac{1}{2}= & 1\frac{1}{2} \\
-2c= & -2\times\frac{1}{2}= & -1 \\
5c= & 5\times\frac{1}{2}= & 2\frac{1}{2} \\
-4c= & -4\times\frac{1}{2}= & -2 \\
\hline
2c= & 2\times\frac{1}{2}= & 1
\end{array}$$

2. What is the sum of $7ab-3ab+2ab+5ab-10ab$?

OPERATION.

$$\begin{array}{r}
7ab \\
-3ab \\
2ab \\
5ab \\
-10ab \\
\hline
ab
\end{array}$$

If the beginner finds the above too difficult, he may arrange the positive and negative terms in two distinct columns as in this second

OPERATION.

$$\begin{array}{r@{}l} 7ab- & \;\;3ab \\ 2ab- & 10ab \\ 5ab & \\ \hline 14ab- & 13ab=ab \end{array}$$

3. What is the sum of $3ax-2ax+5ax-7ax+4ax$?

Ans. $3ax$.

4. What is the sum of $11bc-10bc+17bc-bc$? *Ans.* $17bc$.

5. What is the sum of $49axy-37axy-10axy+100axy-6axy+4axy$? *Ans.* $100axy$.

6. What is the sum of $6pq+3pq-7pq+8pq-pq$?

Ans. $9pq$.

7. What is the sum of $3mg-mg+6mg-8mg+10mg$?

Ans. $10mg$.

8. What is the value of $10h-h+6h-20h+8h$, when $h=5$?

Ans. $3h=3\times5=15$.

9. What is the value of $4x-10x+x-6x+5x$, when $x=4$?

Ans. $-6x=-6\times4=-24$.

10. What is the value of $3ad-ad+6ad-12ad$, when $a=2$, and $d=3$? *Ans.* $-4ad=-4\times2\times3=-24$.

CASE II.

(19.) *When both quantities and signs are different.*

Suppose we wish to find the sum of $3a-5b+6a+2b-7a-b+4a+8b$.

We first proceed to find the sum of the a's by Case I. Thus,

$$\begin{array}{r} 3a \\ 6a \\ -7a \\ 4a \\ \hline 6a \end{array} = \text{the sum of the } a\text{'s.}$$

In the same way we find the sum of the b's, as follows:

$$\begin{array}{rl} -5b & \\ 2b & \\ -\ b & \\ 8b & \\ \hline 4b & =\text{the sum of the } b\text{'s.} \end{array}$$

Connecting these two results with their proper signs, we get $6a+4b$ for the sum sought. We must keep in mind, that different letters cannot be united into one single quantity or term. $6a$ and $4b$ cannot make either $10a$ or $10b$, any more than 6 apples and 4 butternuts can make 10 apples or 10 butternuts.

From this we see that examples under this Case may be wrought by the following

RULE.

I. Find the respective sums of like terms as in Case I.

II. Then write these sums one after another, with their proper signs.

EXAMPLES.

1. What is the sum of $3ax-2ab+4xy-2ax+3xy+7ab-2xy+6ax$?

$$\begin{array}{rl} 3ax & \\ -2ax & \\ 6ax & \\ \hline 7ax & =\text{sum of the terms containing } ax. \\ \hline \end{array}$$

$$\begin{array}{rl} -2ab & \\ 7ab & \\ \hline 5ab & =\text{sum of the terms containing } ab. \\ \hline \end{array}$$

$$\begin{array}{rl} 4xy & \\ 3xy & \\ -2xy & \\ \hline 5xy & =\text{sum of the terms containing } xy. \\ \hline \end{array}$$

Therefore, $7ax+5ab+5xy=$the total sum.

2. What is the sum of $b-8x-y+8x+6y-4y-7b+3y-3x+10x-y+9b-x+3b+8b$?

By arranging the corresponding terms under each other, we obtain the following

OPERATION.

$$\begin{array}{rr}
 & b-8x-y \\
 & -7b+8x+6y \\
 & 9b-3x-4y \\
 & 3b+10x+3y \\
 & 8b-x-y \\
\hline
\textit{Ans.} & 14b+6x+3y \\
\hline
\end{array}$$

3.

$$\begin{array}{rr}
 & 4a^2+5an \\
 & -3a^2-7an \\
 & 2a^2-3an \\
 & 5a^2+10an \\
\hline
\textit{Ans.} & 8a^2+5an \\
\hline
\end{array}$$

4.

$$\begin{array}{rr}
 & 4am-3am^2-6ab \\
 & -7am+4am^2+ab \\
 & -8am-10am^2-6ab \\
 & am+am^2+20ab \\
\hline
\textit{Ans.} & -10am-8am^2+9ab \\
\hline
\end{array}$$

5.

$$\begin{array}{rr}
 & 2a^2x-3ax^2+2ab \\
 & -7a^2x+4ax^2-8ab \\
 & -6a^2x+10ax^2+12ab \\
\hline
\textit{Ans.} & -11a^2x+11ax^2+6ab \\
\hline
\end{array}$$

6.

$$\begin{array}{rr}
 & 3a^2b^3-7ab^4+5axy \\
 & -7a^2b^3-2ab^4-axy \\
 & 8a^2b^3+ab^4-7axy \\
\hline
\textit{Ans.} & 4a^2b^3-8ab^4-3axy \\
\hline
\end{array}$$

7. What is the sum of $3ah+6am-9xy+3ab-xy+4ah+10am-7xy-6ab+5xy+4ah-13am$?

Ans. $11ah+3am-12xy-3ab$.

8. What is the sum of $x^2-y^3+4xy-7x^2+8y^3-10xy+5x^2+2y^3+12xy$? *Ans.* $-x^2+9y^3+6xy$.

9. What is the sum of $3af-2g^2+7x-af+5g^2-10x+x-4af+3g^2-7x+h^4+x-3h^4$? *Ans.* $-2af+6g^2-8x-2h^4$.

10. What is the sum of $3an^2-6a^3n+5a^2n^2-a^3n+7an^2-6a^3n+7a^2n^2-10an^2+3a^3n+8an^2$? *Ans.* $8an^2-10a^3n+12a^2n^2$.

11. What is the sum of $3ax+5bx+7cx-ax-2bx-3cx+10ax-12bx+6bx-11cx+4ax+6bx-cx$? *Ans.* $16ax+3bx-8cx$.

12. What is the sum of $7r^2+3ry+y^2-r^2-4ry+6y^2-3r^2-7ry+5y^2+5r^2-ry+y^2-7ry$? *Ans.* $8r^2-16ry+13y^2$.

13. What is the sum of $8m^2-mn+6n^2-m^2+3mn-6n^2+4mn-4m^2-5n^2+6m^2-6mn$? *Ans.* $9m^2-5n^2$.

14. What is the sum of $16p^3+11p^2q-5pq^2-q^3+p^3-p^2q+10pq^2+q^3-17p^3+8pq^2-4p^2q$? *Ans.* $6p^2q+13pq^2$.

15. What is the sum of $8amx-4xy+5y^2+9amx+10xy-11y^2+amx+5xy+17y^2-3amx$? *Ans.* $15amx+11xy+11y^2$.

16. What is the sum of $3a-4bc-xy+17a-13bc-4xy+8bc+10a-3xy+18+xy$? *Ans.* $30a-9bc-7xy+18$.

17. What is the sum of $x+y+z-7x-3z+8y-z+4y+8x-7z-11y+12z$? *Ans.* $2x+2y+2z$.

18. What is the sum of $3k-4l-7m+2l-10k-m+10l+12k-6+l+6$? *Ans.* $5k+9l-8m$.

19. Find the sum of $15r-11s-13t-17r+19s+23t+29r+31s+37t$. *Ans.* $27r+39s+47t$.

20. Find the sum of $u-v+w-3+4w+5v-7+10u+3w+5v+w$. *Ans.* $11u+9v+9w-10$.

21. Find the sum of $1\frac{1}{2}x+3\frac{1}{4}y+5\frac{1}{6}z+8+7x+2\frac{3}{4}y-\frac{1}{2}x-\frac{1}{6}z-2$. *Ans.* $8x+6y+5z+6$.

22. Find the sum of $a^2+b^4+c^3-4a^2-5c^3-7b^4+8a^2+10c^3-12a+3b+4a-5b$. *Ans.* $5a^2-6b^4+6c^3-8a-2b$.

23. Find the sum of $3ab^2-5a^2b+8a^2b^2-8ab+10ab^2+12ab+6ab^2+8a^2b-a^2b^2$. *Ans.* $19ab^2+3a^2b+7a^2b^2+4ab$.

24. Collect into one sum the following: $a^3b^2c+3a^2b^3c-5ab^3c^2-4ab^2c^3+6a^3b^2c+7a^2b^3c-2ab^2c^3-ab^3c^2-a^3b^2c$.

Ans. $6a^3b^2c+10a^2b^3c-6ab^3c^2-6ab^2c^3$.

25. Collect into one sum the following: $4x^4-3y^3-2z^2+4z^2+5y^3+8+16x^4-10+z^2-y^3+x^4-3$. *Ans.* $21x^4+y^3+3z^2-5$.

26. Collect into one sum the following: $4\frac{1}{2}x^3-3\frac{1}{3}y^2-2z+5\frac{1}{2}x^3+4\frac{1}{2}-3x^3+2\frac{1}{3}y^2-5z+4y^2-5x^3-9\frac{1}{2}$.

Ans. $2x^3+3y^2-7z-5$.

27. Collect into one sum the following: $3w^2xy-2wx^2y+3wxy^2+5w^2xy+5wx^2y-7wxy^2+8w^2xy-3wx^2y-6wxy^2$.

Ans. $16w^2xy-10wxy^2$.

SUBTRACTION.

(20.) SUBTRACTION, in Algebra, is finding the simplest expression for the difference of two algebraic expressions.

If we wish to subtract b from a, we obviously obtain $a-b$ (Art. 5), which is the same as the addition of a and $-b$. (See Rule under Art. 19.)

Again, if we wish to subtract $b-c$ from a, we first subtract b from a and find, as above, $a-b$ for the result. Now, it is obvious that we have subtracted too much by the quantity c, therefore adding c to the above result, we finally obtain $a-b+c$, which is the same as the addition of a and $-b+c$.

From this we conclude that subtracting a quantity is the same as adding it after the signs have been changed.

Hence, for the subtraction of algebraic quantities, we have this

RULE.

I. Write the terms to be subtracted, under the similar terms, if there are any, of those from which they are to be subtracted.

II. Conceive the signs of the terms of the polynomial which is to be subtracted, to be changed, and then proceed as in addition.

EXAMPLES.

1. From $4a+3b-2c$, subtract $a+2b+c$.

Actually changing the signs of the polynomial to be subtracted, and then placing it under the other polynomial, and proceeding as in addition, we have this

OPERATION.

$$\begin{array}{r l} 4a+3b-2c & \\ -a-2b-\ c & \\ \hline 3a+\ b-3c & \text{difference sought.} \\ \hline \end{array}$$

In practice we do not usually change the signs, but only conceive them to be changed, before performing our addition.

2.

$$\begin{array}{l r} \text{From} & 32b+\ 3a \\ \text{Take} & 5b+18a \\ \hline \text{Remainder} & 27b-15a \\ \hline \end{array}$$

3.

$$\begin{array}{l r} \text{From} & 6aby-3xy \\ \text{Take} & -2aby+2xy \\ \hline \text{Remainder} & 8aby-5xy \\ \hline \end{array}$$

4.

$$\begin{array}{l c} \text{From} & x^2+2xy+y^2 \\ \text{Take} & x^2-2xy+y^2 \\ \hline \text{Rem.} & 4xy \\ \hline \end{array}$$

5.

$$\begin{array}{l r} \text{From} & 13x^2y-4xy^2+\ y^3 \\ \text{Take} & 6x^2y+\ xy^2-\ y^3 \\ \hline \text{Rem.} & 7x^2y-5xy^2+2y^3 \\ \hline \end{array}$$

6. From $17abc+b^3-3bc$, subtract $abc-6b^3-4bc$.

Ans. $16abc+7b^3+bc$.

7. From $11a^2x-13ax^2+17bx^2$, subtract $10a^2x-15ax^2-19bx^2$.

Ans. $a^2x+2ax^2+36bx^2$.

8. From $4my-3nx+5mn$, subtract $3my-4nx-5mn$.

Ans. $my+nx+10mn$.

9. From $9gh^2-7gy+5hm^2$, subtract $3gh^2+5gy+hm^2$.
Ans. $6gh^2-12gy+4hm^2$.

10 From $11a^4x^3y+3ax-5ab+mg-3x^2y^2$, take $5xy+4a^4x^3y-6ax+9ab$. *Ans.* $7a^4x^3y+9ax-14ab+mg-3x^2y^2-5xy$.

11. From $3a^2m-6x^2y^3+2xy$, take $4a^2m+6x^2y^3+5xy$.
Ans. $-a^2m-12x^2y^3-3xy$.

12. From $3a^2bc-7axy+3my+a$, take $a^2bc+8axy+6a-4my$.
Ans. $2a^2bc-15axy+7my-5a$.

13. From $4ab^3-3a^3b+5m^3+6n^2$, take $ab^3+a^3b-m^3+n^2$.
Ans. $3ab^3-4a^3b+6m^3+5n^2$.

14. From $3w^2xy+8wx^2y-7wxy^2$, take $w^2xy-wx^2y-9wxy^2$.
Ans. $2w^2xy+9wx^2y+2wxy^2$.

15. From $4\frac{1}{2}x^3-3\frac{1}{3}y^2+4\frac{1}{4}z^4$, take $1\frac{1}{2}x^3-1\frac{1}{3}y^2+1\frac{1}{4}z^4+a-b$.
Ans. $3x^3-2y^2+3z^4-a+b$.

16. From $3a^2b-2ab^2+4ab-3a^2b^2$, take $3ab+a^2b^2-a^2b-3ab^2-m+n$. *Ans.* $4a^2b+ab^2+ab-4a^2b^2+m-n$.

17. Take $2xy-3xy^2+xz-4yz$ from $4xy+3xy^2-xz-5yz+16$. *Ans.* $2xy+6xy^2-2xz-yz+16$.

18. Take $a+b^2-c^3+x^2-y^3$ from $7x^2-8y^3-a^2+3a$.
Ans. $2a-b^2+c^3+6x^2-7y^3-a^2$.

(21.) We can express the subtraction of one polynomial from another, by writing the polynomial which is to be subtracted, after inclosing it in a parenthesis, immediately after the other polynomial from which it is to be subtracted, observing to place the negative sign before the parenthesis.

Thus, $ab-6xy+3am-(-3ab+xy-am)$, denotes that the polynomial inclosed in the parenthesis is to be subtracted from the one which precedes it; and since, to perform subtraction, we must change all the signs of the terms to be subtracted, it follows that the parenthesis may be removed, provided we change the signs of all the terms which it incloses.

The above expression will thus become

$$ab-6xy+3am+3ab-xy+am,$$

which, reduced by uniting terms having like letters, gives

$$4ab-7xy+4am.$$

In the same way we have

$$a^2b+xy-7am-(mx+6-13x^2), \text{ equivalent to}$$
$$a^2b+xy-7am-mx-6+13x^2.$$

Also, $4axy-3ay^2-7x^3-(13a-my+6)$, is the same as

$$4axy-3ay^2-7x^3-13a+my-6.$$

It is also obvious that we may inclose any number of terms of a polynomial, within a parenthesis, with a negative sign before it, if we observe to change the signs of all the terms thus inclosed.

In this way, the polynomial

$$a^2b+xy-7am-mx-6+13x^2,$$

is made to take the following successive forms:

$$a^2b+xy-7am-(mx+6-13x^2)$$
$$a^2b+xy-(7am+mx+6-13x^2)$$
$$a^2b-(-xy+7am+mx+6-13x^2)$$
$$a^2b+xy-7am-mx-(6-13x^2).$$

In a similar manner we have

$$2x^2+3y^2-7z-5+a^2-b^4,$$

equivalent to the following expressions:

$$2x^2+3y^2-7z-5-(-a^2+b^4)$$
$$2x^3+3y^2-7z-(5-a^2+b^4)$$
$$2x^3+3y^2-(7z+5-a^2+b^4)$$
$$2x^2-(-3y^2+7z+5-a^2+b^4).$$

MULTIPLICATION.

(22.) If we wish to multiply a by b, we must repeat a as many times as there are units in b, which, by (Art. 7), is done by writing b immediately after a. Thus, a multiplied by b is ab.

Again, if we wish to multiply a by $-b$, we observe that this is the same as to multiply $-b$ by a; hence, we must repeat $-b$ as many times as there are units in a. Repeating a minus quantity once, twice, thrice, or any number of times, cannot change it to a positive quantity. Therefore, $-b$, multiplied by a, or, which is the same, a multiplied by $-b$, gives $-ab$.

Finally, if we wish to multiply $a-b$ by $c-d$, we first multiply $a-b$ by c; thus, a multiplied by c gives ac, and $-b$ multiplied by c gives $-bc$; or the work may be written as follows:

$$\begin{array}{l} a-b \\ c \\ \hline \underline{ac-bc}=(a-b) \text{ repeated } c \text{ times.} \end{array}$$

This result is evidently too great by the product of $(a-b)$ by d, since it was required to repeat $(a-b)$ as many times as there are units in c less d.

Repeating $(a-b)$ as many times as there are units in d, we have

$$\begin{array}{l} a-b \\ d \\ \hline \underline{ad-bd}=(a-b) \text{ repeated } d \text{ times.} \end{array}$$

Subtracting this last result from the former, we have (Art. 21) $ac-bc-(ad-bd)$, which becomes $ac-bc-ad+bd=(a-b)$ repeated $(c-d)$ times.

Hence, we see that $-b$, when multiplied by $-d$, produces in the product $+bd$.

(23.) *From all this, we discover that the product will have the sign* $+$, *when both factors have like signs, and the product will have the sign* $-$, *when the factors have contrary signs.*

Perhaps the following method may assist the pupil in comprehending the above rule for the sign of the product of two algebraic factors;

We will multiply -7 successively by 4, 3, 2, 1, 0, -1, -2, -3, -4; each multiplier being a unit less than the preceding one.

$$\begin{aligned}-7\times4&=-28\\-7\times3&=-21\\-7\times2&=-14\\-7\times1&=-\ \ 7\end{aligned}$$

If we now examine these results, which have been obtained thus far without any difficulty, we shall notice that the successive products -28, -21, -14, -7, increase regularly by 7. There is no reason why this regular increase of successive products should not continue, how far soever the process of multiplying may be extended. Hence, multiplying by 0, which is 1 less than our last multiplier, we must obtain 7 more than -7, that is, we must obtain 0.

Again, multiplying by -1, which is 1 less than 0, we must obtain 7 more than 0, that is, $+7$. And when we multiply by -2, which is 1 less than -1, we shall have 7 more than the preceding product, $+7$, that is, we shall have $+14$.

Exhibiting the whole process in connection, we have

$$\begin{aligned}-7\times\ \ 4&=-28\\-7\times\ \ 3&=-21\\-7\times\ \ 2&=-14\\-7\times\ \ 1&=-\ \ 7\\-7\times\ \ 0&=\ \ \ \ 0\\-7\times-1&=+\ \ 7\\-7\times-2&=+14\\-7\times-3&=+21\\-7\times-4&=+28\end{aligned}$$

From this we see that the product is *negative* when one of the factors is positive and the other negative.

The product is *positive* when both factors are negative.

The product is *zero* when one of the factors is zero.

(24.) If we wish to multiply $3a^2b$ by $4a^3b^2$, we observe that (Art. 11),

$$3a^2b = 3aab$$
$$4a^3b^2 = 4aaabb.$$

Hence, the product of $3a^2b$ by $4a^3b^2$, will be

$$3aab \times 4aaabb = 12aaaaabbb = 12a^5b^3.$$

From which we discover that the exponent of a, in the product, is equal to the sum of the exponents of a in the factors; likewise, the exponent of b, in the product, is equal to the sum of the exponents of b in the factors.

Hence, the product of several letters having different exponents is equal to the product of all the letters having for exponents the sums of their respective exponents in the factors.

CASE I.

(25.) From what has been said, we have for multiplying together two monomials, this

RULE.

I. Multiply together the coefficients, observing to prefix the sign + when both factors have like signs; and the sign — when they have contrary signs.

II. Write the letters one after another; if the same letter occur in both factors, add their exponents for a new exponent. It must be recollected that when no exponent is expressed, 1 *is understood to be the exponent* (Art. 11).

(26.) As in Arithmetic, it is obvious the product will be the same, in whatever order the letters are placed, but it will be found more convenient in practice, to have a uniform order for their arrangement. The order usually adopted, is, to place them alphabetically

EXAMPLES.

1. Multiply $11ax^3y$ by $3a^4y^3$.	*Ans.* $33a^5x^3y^4$.
2. Multiply $7ab$ by $3ac$.	*Ans.* $21a^2bc$.
3. What is the product of $12xyz$ by $10x$?	*Ans.* $120x^2yz$.
4. What is the product of $8mn$ by $4m^2n^2$?	*Ans.* $32m^3n^3$.
5. What is the product of $-3a^2y$ by $6ag$?	*Ans.* $-18a^3gy$.
6. What is the product of $12ar$ by $-3abc$?	*Ans.* $-36a^2bcr$.
7. What is the product of $-6x^3y$ by $-7xy^3$?	*Ans.* $42x^4y^4$.
8. What is the product of $-10am^2n^4$ by -11?	*Ans.* $110am^2n^4$.
9. What is the product of $-\frac{1}{2}axy$ by $8ay$?	*Ans.* $-4a^2xy^2$.
10. Multiply $-\frac{1}{2}ax$ by $-\frac{1}{4}ay$.	*Ans.* $\frac{1}{8}a^2xy$.
11. Multiply $-\frac{1}{5}gm$ by $-\frac{1}{6}m^2$.	*Ans.* $\frac{1}{30}gm^3$.
12. Multiply $\frac{3}{8}ax$ by $\frac{1}{5}ax^2z$.	*Ans.* $\frac{3}{40}a^2x^3z$.
13. Multiply $7m^9n^8p^7$ by $6mn^2p^3$.	*Ans.* $42m^{10}n^{10}p^{10}$.
14. Multiply $3x$ by $-7xy^2$.	*Ans.* $-21x^2y^2$.
15. Multiply $-15abc$ by $-2a^2bc^3$.	*Ans.* $30a^3b^2c^4$.
16. Multiply $-\frac{6}{7}agm$ by $\frac{3}{5}axy$.	*Ans.* $-\frac{18}{35}a^2mgxy$.
17. Multiply $\frac{3}{4}gh$ by $-\frac{5}{7}g^2h^3$.	*Ans.* $-\frac{15}{28}g^3h^4$.

CASE II.

(27.) It is obvious that polynomials may be multiplied by the following

RULE.

I. Multiply all the terms of the multiplicand, successively by each term of the multiplier, and observe the same rules for the signs and exponents as in Case I.

II. When there arise several partial products alike, they must be placed under each other and then added together in the total product.

(28.) The total product will be the same in whatever order we multiply by the terms of the multiplier, but for the sake of order and uniformity, we begin with the left-hand term. By this means we are always extending our work towards the right, which is more natural and simple, than to commence on the right, and extend the work towards the left, as we are compelled to do in arithmetical multiplication.

EXAMPLES.

1. What is the product of $3a^2-6ax$ by $3a-m$?

OPERATION.

$3a^2-6ax$ multiplicand.
$3a-m$ multiplier.
$9a^3-18a^2x-3a^2m+6amx=$total product.

EXPLANATION.

Having placed the multiplier under the multiplicand, we commence with $3a$, the left-hand term of the multiplier, and multiply it into $3a^2$, the left-hand term of the multiplicand, and obtain the product $9a^3$, which we place for the left-hand term of the product. We then multiply $3a$ into $-6ax$, the second term of the multiplicand, and obtain the product $-18a^2x$, which constitutes the second term of the product. Having multiplied all the terms of the multiplicand by $3a$, we next multiply by $-m$. Thus, $-m$ multiplied into $3a^2$, gives $-3a^2m$, which is the third term of the product, and $-m$ multiplied into $-6ax$, gives $6amx$, which is the last term of the product.

2. What is the product of $6x^2-3y^3$ by x^2-2y^3?

OPERATION.

$$\begin{array}{lllll}
6x^2 & -\;3y^3 & & & \text{multiplicand.}\\
x^2 & -\;2y^3 & & & \text{multiplier.}\\
\hline
6x^4 & -\;3x^2y^3 & & & \text{product by } x^2.\\
 & -12x^2y^3 & +6y^6 & & \text{product by } -2y^3.\\
\hline
6x^4 & -15x^2y^3 & +6y^6 & = & \text{total product.}
\end{array}$$

3. Multiply $a+b$ by $a+b$.

OPERATION.

$$\begin{array}{rllll}
 & a & +b & & \\
 & a & +b & & \\
\hline
 & a^2 & +ab & & \\
 & & +ab & +b^2 & \\
\hline
\textit{Ans.} & a^2 & +2ab & +b^2 &
\end{array}$$

4. Multiply $a-b$ by $a-b$.

OPERATION.

$$\begin{array}{rllll}
 & a & -\;b & & \\
 & a & -\;b & & \\
\hline
 & a^2 & -ab & & \\
 & & -ab & +b^2 & \\
\hline
\textit{Ans.} & a^2 & -2ab & +b^2 &
\end{array}$$

5. Multiply $a+b$ by $a-b$.

OPERATION.

$$\begin{array}{lccccc}
 & a & + & b & & \\
 & a & - & b & & \\ \hline
 & a^2 & + & ab & & \\
 & & - & ab & - & b^2 \\ \hline
\text{Ans.} & a^2 & & & - & b^2 \\ \hline
\end{array}$$

Examples 3, 4, and 5 may take the following forms:

$$(a+b)\,(a+b)=(a+b)^2=a^2+b^2+2ab.$$
$$(a-b)\,(a-b)=(a-b)^2=a^2+b^2-2ab.$$
$$(a+b)\,(a-b)=\qquad a^2-b^2.$$

Translating these into arithmetical language, they become

I. The square of the sum of two numbers, is equal to the sum of their squares increased by twice their product.

II. The square of the difference of two numbers, is equal to the sum of their squares diminished by twice their product.

III. The sum of two numbers multiplied by their difference, is equal to the difference of their squares.

Under Chapter VII. we have derived many interesting propositions, by translating algebraic formula into common language.

6. What is the product of b^2m-3ay by $6x-3$?

Ans. $6b^2mx-18axy-3b^2m+9ay$.

7. What is the product of $a^6+a^4+a^2$ by a^2-1?

Ans. a^8-a^2.

8. Multiply a^2+ax+x^2 by a^2-ax+x^2.

Ans. $a^4+a^2x^2+x^4$.

9. What is the product of $7m+y$ by $7m-y$?

Ans. $49m^2-y^2$.

10. What is the product of $a+b+c$ by $a+b+c$?

Ans. $a^2+2ab+2ac+b^2+2bc+c^2$.

11. What is the product of $\frac{1}{2}a^2-3x$ by $\frac{1}{3}b-2y$?
Ans. $\frac{1}{6}a^2b-bx-a^2y+6xy$.

12. What is the product of $x^4-x^3+x^2-x+1$ by x^2+x-1?
Ans. $x^6-x^4+x^3-x^2+2x-1$.

13. Multiply $a^3+a^2b+ab^2+b^3$ by $a-b$. *Ans.* a^4-b^4.

14. What is the product of $2x^2+5x-7$ by $2x-3$?
Ans. $4x^3+4x^2-29x+21$.

15. What is the product of $a-b-c$ by $a+b-c$?
Ans. $a^2-2ac+c^2-b^2$.

16. What is the product of $x^2+2xy+y^2$ by $x^2-2xy+y^2$?
Ans. $x^4-2x^2y^2+y^4$.

17. Multiply x^3+x^2+x+1 by $x-1$. *Ans.* x^4-1.

18. Multiply $x^4+x^3+x^2+x+1$ by $x-1$. *Ans.* x^5-1.

19. Multiply $x^5+x^4+x^3+x^2+x+1$ by $x-1$. *Ans.* x^6-1.

20. Multiply $x^6+x^5+x^4+x^3+x^2+x+1$ by $x-1$.
Ans. x^7-1.

21. Multiply x^2-x+1 by $x+1$. *Ans.* x^3+1.

22. Multiply x^3-x^2+x-1 by $x+1$. *Ans.* x^4-1.

23. Multiply $x^4-x^3+x^2-x+1$ by $x+1$. *Ans.* x^5+1.

24. Multiply $x^5-x^4+x^3-x^2+x-1$ by $x+1$. *Ans.* x^6-1.

DIVISION.

(29.) We know by the principles of Arithmetic, that if in division, we multiply the divisor into the quotient, the product will be the dividend.

Therefore, referring to what has been said under Multiplication (Art. 23), we infer that when the dividend has the sign +, the divisor and quotient must have the same sign; but when the dividend has the sign —, then the divisor and quotient must have contrary signs.

(30.) *Hence, when the dividend and divisor have like signs, the quotient will have the sign +, and when the dividend and divisor have contrary signs, the quotient will have the sign —.*

We have also seen under Multiplication (Art. 24), that the product of several letters of different powers is equal to the product of all the letters having for exponents the sum of their respective exponents.

(31.) *Hence, if we divide any power of a letter by a different power of the same letter, it is obvious that the quotient will be a power of the same letter having for an exponent the excess of the exponent in the dividend above that of the divisor.*

(32.) If we divide $a^5 = aaaaa$, continually by a, we shall obtain the following results:

$$a^5 \div a = a^{5-1} = a^4 = aaaa$$

$$a^4 \div a = a^{4-1} = a^3 = aaa$$

$$a^3 \div a = a^{3-1} = a^2 = aa$$

$$a^2 \div a = a^{2-1} = a^1 = a$$

$$a^1 \div a = a^{1-1} = a^0 = 1 \quad \left\{ \text{since the divisor is equal to the dividend.} \right.$$

$$a^0 \div a = a^{0-1} = a^{-1} = \frac{1}{a} = \text{reciprocal of } a \text{ (Art. 12).}$$

$$a^{-1} \div a = a^{-1-1} = a^{-2} = \frac{1}{aa} = \frac{1}{a^2} = \text{reciprocal of } a^2.$$

$$a^{-2} \div a = a^{-2-1} = a^{-3} = \frac{1}{aaa} = \frac{1}{a^3} = \text{reciprocal of } a^3.$$

$$a^{-3} \div a = a^{-3-1} = a^{-4} = \frac{1}{aaaa} = \frac{1}{a^4} = \text{reciprocal of } a^4.$$

$$a^{-4} \div a = a^{-4-1} = a^{-5} = \frac{1}{aaaaa} = \frac{1}{a^5} = \text{reciprocal of } a^5.$$

(33.) *From the above scheme, we see that whenever the exponent of a quantity becomes* 0, *the value of the quantity is reduced to* 1.

(34.) *That whenever the exponent of a quantity is negative, the quantity is the reciprocal of what it would be were its exponent positive.*

(35.) *Hence, changing the sign of the exponent of a quantity, is the same as taking the reciprocal of the quantity.*

CASE I.

(36.) From what has been said, we have, for dividing one monomial by another, this

RULE.

I. Divide the coefficient of the dividend by that of the divisor, observing to prefix to the quotient the sign +, *when the signs of the dividend and divisor are alike; and the sign* −, *when they have contrary signs.*

II. Subtract the exponents of the letters in the divisor from the exponents of the corresponding letters in the dividend; if letters occur in the divisor which do not in the dividend, they may (Art. 35) *be written in the quotient by changing the signs of their exponents.*

(37.) It must be recollected here, and in all cases hereafter, that when the exponent of a letter is not written, 1 is always understood (Art. 11), and when the exponent becomes 0, the value of the power is 1 (Art. 33).

EXAMPLES.

1. Divide $15a^3b$ by $5abxy^2$.

OPERATION.

$$\frac{15a^3b}{5abxy^2}=3a^2x^{-1}y^{-2}.$$

EXPLANATION.

We first divide 15 by 5, and obtain 3 for the quotient, which we consider as positive, since the signs of the dividend and divisor were alike, each being positive. The exponent of a in the divisor is 1 understood (Art. 37), which subtracted from 3, the exponent of a in the dividend, gives 2 for the exponent of a in the quotient. The exponent of b in the denominator, is the same as the exponent of b in the numerator, each being 1; hence, the exponent of b in the quotient is 0; but when the exponent of a quantity becomes 0, its value is 1 (Art. 33): so that b does not appear in the quotient. Since the letters x and y do not occur in the dividend, we write the x and y in the quotient with the signs of their exponents changed.

This quotient is read: "Three times the square of a into the reciprocal of x, into the reciprocal of the square of y."

2. Divide $6a^4b^2$ by $3ab$. *Ans.* $2a^3b$.
3. Divide $12x^4y^5z^6$ by $4xyz$. *Ans.* $3x^3y^4z^5$.
4. Divide $10m^3xz^6$ by $5x$. *Ans.* $2m^3z^6$.
5. Divide $-14a^3b^2$ by $7ab^2$. *Ans.* $-2a^2$.
6. Divide $-16a^5b^4c^3$ by $-8ab^2c^3$. *Ans.* $2a^4b^2$.
7. Divide $-13a^3y^4$ by $-26ay$. *Ans.* $\frac{1}{2}a^2y^3$.
8. Divide $7l^3m^4n$ by $-3ln$. *Ans.* $-\frac{7}{3}l^2m^4$.
9. Divide $8ab^2d^4$ by $4a^4bd$. *Ans.* $2a^{-3}bd^3$.
10. Divide $20m^3n^2p$ by $10m^4n^3$. *Ans.* $2m^{-1}n^{-1}p$.
11. Divide $35xyz$ by $7x^2y^3$. *Ans.* $5x^{-1}y^{-2}z$.
12. Divide $-40x^3y$ by $4ab^2x$. *Ans.* $-10a^{-1}b^{-2}x^2y$.
13. Divide $3xy$ by $-6x^2$. *Ans.* $-\frac{1}{2}x^{-1}y$.
14. Divide $-4m^2y$ by $8m^4$. *Ans.* $-\frac{1}{2}m^{-2}y$.

The results of the six preceding operations are read as follows:

9. "Twice the reciprocal of the cube of a, into b, into the cube of d."

10. "Twice the reciprocal of m, into the reciprocal of n, into p."

11. "Five times the reciprocal of x, into the reciprocal of the square of y, into z."

12. "Minus ten times the reciprocal of a, into the reciprocal of the square of b, into the square of x, into y."

13. "Minus one-half of the reciprocal of x, into y."

14. "Minus one-half of the reciprocal of the square of m, into y."

(38.) To divide one polynomial by another, we shall imitate the arithmetical method of *long division*. And in the arrangement of the work, we shall follow the *French* method of placing the divisor at the right of the dividend. Thus, to divide

$$a^3+a^2x+ab+bx \text{ by } a+x,$$

we proceed as follows:

OPERATION.

$$\begin{array}{r|l}
\text{Dividend}=a^3+a^2x+ab+bx & a\ +x=\text{divisor.} \\ \cline{2-2}
\underline{a^3+a^2x\qquad\qquad} & a^2+b=\text{quotient.} \\
ab+bx & \\
\underline{ab+bx} & \\
\underline{\qquad 0} &
\end{array}$$

EXPLANATION.

Having placed the divisor at the right of the dividend, we seek how many times its left-hand term is contained in the left-hand term of the dividend, which we find to be a^2, which we place directly under the divisor, and then multiply the divisor by it, and subtract the product from the dividend; then bringing down the remaining terms, we again seek how many times the left-hand term of the divisor is contained in the left-hand term of this remainder, which we find to be b; we then multiply the divisor by b, and again subtracting, there remains nothing; so that a^2+b is the complete quotient.

(39.) That the operation may be the most simple, it will be necessary to arrange both dividend and divisor according to the powers of some particular letter, commencing with the highest power.

CASE II.

(40.) To divide one polynomial by another, we have this

RULE.

I. Arrange the dividend and divisor with reference to a certain letter; then divide the first term on the left of the dividend by the first term on the left of the divisor; the result is the first term of the quotient multiply the divisor by this term, and subtract the product from the dividend.

II. Then divide the first term of the remainder by the first term of the divisor, which gives the second term of the quotient; multiply the divisor by this second term; and subtract the product from the result after the first operation. Continue this process until we obtain 0 *for remainder; or when the division does not terminate, which is frequently the case, we can carry on the above process as far as we choose, and then place the last remainder over the divisor, forming a fraction, which must be added to the quotient.*

EXAMPLES.

1. What is the quotient of $2a^2b+b^3+2ab^2+a^3$ divided by a^2+b^2+ab?

Arranging the terms according to the powers of a, and proceeding agreeably to the above rule, we have this

OPERATION.

$$\begin{array}{r|l}
\text{Dividend}=a^3+2a^2b+2ab^2+b^3 & a^2+ab+b^2=\text{divisor.} \\ \cline{2-2}
\underline{a^3+\ a^2b+\ ab^2\quad} & a+b=\text{quotient.} \\
a^2b+\ ab^2+b^3 & \\
\underline{a^2b+\ ab^2+b^3} & \\
\underline{\qquad 0} &
\end{array}$$

2. What is the quotient of $4x^3+4x^2-29x+23$ by $2x-3$?

OPERATION.

$$\begin{array}{r|l}
4x^3+\ 4x^2-29x+23 & 2x-3 \\ \cline{2-2}
4x^3-\ 6x^2\qquad\qquad & \\
\overline{10x^2-29x\quad} & 2x^2+5x-7+\dfrac{2}{2x-3} \\
\underline{10x^2-15x\quad} & \\
-14x+23 & \\
\underline{-14x+21} & \\
\underline{2}=\text{remainder.} &
\end{array}$$

In this example, we find 2 for remainder, which, being placed over the divisor $2x-3$, gives the fraction $\frac{2}{2x-3}$ to be added to our quotient.

3. Divide $a^4+a^2z^2+z^4$ by a^2+az+z^2.

OPERATION.

$$\begin{array}{r|l}
a^4+a^2z^2+z^4\quad & a^2+az+z^2 \\ \cline{2-2}
\underline{a^4+a^3z\ +a^2z^2} & a^2-az+z^2 \\
-a^3z\ +z^4\qquad\qquad & \\
\underline{-a^3z\ -a^2z^2-az^3} & \\
a^2z^2+az^3+z^4 & \\
\underline{a^2z^2+az^3+z^4} & \\
\underline{\qquad 0} &
\end{array}$$

4. Divide $a^3-3a^2b+3ab^2-b^3$ by $a-b$.

Ans. $a^2-2ab+b^2$.

5. Divide $25a^2-10ax+x^2$ by $5a-x$. *Ans.* $5a-x$.

6. Divide $6a^4-96$ by $3a-6$. *Ans.* $2a^3+4a^2+8a+16$.

7. Divide $3x^4-26x^3y-14xy^3+37x^2y^2$ by $3x^2-5xy+2y^2$. *Ans.* x^2-7xy.

8. Divide $x^2+2xy+y^2$ by $x+y$. *Ans.* $x+y$.

9. Divide $x^4-4x^3y+6x^2y^2-4xy^3+y^4$ by $x^2-2xy+y^2$. *Ans.* $x^2-2xy+y^2$.

10. Divide $64m^4n^6-25m^2n^8$ by $8m^2n^3+5mn^4$. *Ans.* $8m^2n^3-5mn^4$.

11. Divide m^6-1 by $m+1$. *Ans.* $m^5-m^4+m^3-m^2+m-1$.

12. Divide y^4-z^4 by $y+z$. *Ans.* $y^3-y^2z+yz^2-z^3$.

13. Divide x^5-y^5 by $x-y$. *Ans.* $x^4+x^3y+x^2y^2+xy^3+y^4$.

14. Divide a^4-b^4 by $a^3+a^2b+ab^2+b^3$. *Ans.* $a-b$.

15. Divide 1 by $1+b$.

$$\textit{Ans.}\ 1-\frac{b}{1+b}=1-b+\frac{b^2}{1+b}=1-b+b^2-\frac{b^3}{1+b}$$
$$=1-b+b^2-b^3+\frac{b^4}{1+b}=\&c.$$

This division will never terminate, however far it is extended. Each result is the true quotient, as may be readily proved by multiplying it by the divisor, $1+b$. The product will in each case be 1, the dividend.

We have something analogous to this, in the operation of converting many of our vulgar fractions into their equivalent decimal values. Thus, for instance,

$$\tfrac{1}{3}=0{\cdot}3\tfrac{1}{3}=0{\cdot}33\tfrac{1}{3}=0{\cdot}333\tfrac{1}{3}=\&c.$$

16. Divide $1+b$ by $1-b$.

$$\textit{Ans.}\ 1+\frac{2b}{1-b}=1+2b+\frac{2b^2}{1-}=1+2b+2b^2+\frac{2b^3}{1-b}=\&c$$

17. Divide $1-b$ by $1+b$.

Ans. $1-\frac{2b}{1+b}=1-2b+\frac{2b^2}{1+b}=1-2b+2b^2-\frac{2b^3}{1+b}=$&c.

18. Divide $a^2-2ac+c^2-b^2$ by $a-b-c$.

Ans. $a+b-c$.

19. Divide $4x^3+4x^2-29x+24$ by $2x-3$.

Ans. $2x^2+5x-7+\frac{3}{2x-3}$.

CHAPTER II.

ALGEBRAIC FRACTIONS.

(41.) In our operations upon algebraic fractions, we shall follow the corresponding operations upon numerical fractions, so far as the nature of the subject will admit.

CASE I.

To reduce a monomial fraction to its lowest term, we have this

RULE.

I. Find the greatest common measure of the coefficients of the numerator and denominator. (See Arithmetic.)

II. Then, to this greatest common measure, annex the letters which are common to both numerator and denominator; give to these letters the lowest exponent which they have, whether in the numerator or denominator. The result will be the greatest common measure of both numerator and denominator.

III. Divide both numerator and denominator by this greatest common measure (by rule under Art. 36), *and the resulting fraction will be in the lowest terms.*

EXAMPLES.

1. Reduce $\frac{375a^3bxy}{15ab^2xy^3}$ to its lowest terms.

The greatest common measure of 375 and 15 is 15, to which annexing $abxy$, we have $15abxy$ for the greatest common measure of both numerator and denominator. Dividing the numerator by $15abxy$, we find

$$375a^2bxy \div 15abxy = 25a^2.$$

In the same way we find

$$15ab^2xy^3 \div 15abxy = by^2;$$

hence, we have

$$\frac{375a^2bxy}{15ab^2xy^3} = \frac{25a^2}{by^2},$$

which, by Rule under Art. 36, becomes

$$\frac{25a^2}{by^2} = 25a^2b^{-1}y^{-2}.$$

This is read, "Twenty-five times the square of *a*, divided by *b* times the square of *y*, equals twenty-five times the square of *a*, into the reciprocal of *b*, into the reciprocal of the square of *y*."

2. Reduce $\frac{42ax^3yz^5}{35xy^3z^3}$ to its lowest terms.

In this example, the greatest common measure of the numerator and denominator is $7xyz^3$; hence, dividing both numerator and denominator of our fraction by $7xyz^3$, we find

$$\frac{42ax^3yz^5}{35xy^3z^3} = \frac{6ax^2z^2}{5y^2},$$

which is in its lowest terms

3. Reduce $\frac{-18mnx^2y^3}{72mx^3y^6}$ to its lowest terms. *Ans.* $\frac{-n}{4xy^3}$.

4. What is the simplest form of $\frac{13x^3}{26xy^4}$? *Ans.* $\frac{x^2}{2y^4}$.

5. What is the simplest form of $\frac{108ab^3cd^7}{12abcd}$? *Ans.* $9b^2d^6$.

6. What is the simplest form of $\frac{27abcd}{108a^3b^4m}$? *Ans.* $\frac{cd}{4a^2b^3m}$.

7. What is the simplest form of $\frac{16xy^4}{32xy^6}$? *Ans.* $\frac{1}{2y^2}$

8. What is the simplest form of $\frac{21x^2y^7}{35xy^4}$? *Ans.* $\frac{3xy^3}{5}$

9. Reduce $\frac{100a^3bx^2y}{120a^4xy}$ to the simplest form. *Ans.* $\frac{5bx}{6a}$

10. Reduce $\frac{27a^2b^3c}{81a^3b^2x}$ to the simplest form. *Ans.* $\frac{bc}{3ax}$.

11. Reduce $\frac{33mn^2}{55m^3n}$ to the simplest form. *Ans.* $\frac{3n}{5m^2}$.

12. Reduce $\frac{44a^2b}{66ab^2}$ to lowest terms. *Ans.* $\frac{2a}{3b}$.

13. Reduce $\frac{37xy^3}{111xy}$ to lowest terms. *Ans.* $\frac{y^2}{3}$.

14. Reduce $\frac{45mnx}{9m^2nx^2}$ to lowest terms. *Ans.* $\frac{5}{mx}$.

15. Reduce $\frac{10x^{10}y^9}{14x^8y^{10}}$ to lowest terms. *Ans.* $\frac{5x^2}{7y}$.

16. Reduce $\frac{12xm^4}{16x^2m}$ to lowest terms. *Ans.* $\frac{3m^3}{4x}$.

(42.) To obtain a general rule for reducing a fraction whose numerator, or denominator, or both, are polynomials, it would be necessary to show how to find the greatest common measure of two polynomials, a process which is too complicated and difficult for this place.

There are, however, many polynomial fractions, of which the common measure of their respective numerators and denominators are at once obvious, and of course they are then readily reduced. We will illustrate this by a few

EXAMPLES.

1. Reduce $\frac{3xy+xz}{ax+bx}$ to its simplest form.

In this example, we see that x will divide both the numerator and denominator. Hence,

$$\frac{3xy+xz}{ax+bx}=\frac{3y+z}{a+b}.$$

2. Reduce $\frac{ac+ay}{bc+by}$ to its simplest form.

In this example, the greatest measure of the numerator and denominator is obviously $c+y$. Hence,

$$\frac{ac+ay}{bc+by}=\frac{a}{b}.$$

3. Reduce $\frac{abx+ax}{abx-ax}$ to its simplest form. *Ans.* $\frac{b+1}{b-1}$.

4. Reduce $\frac{x^3-b^2x}{x^2+2bx+b^2}=\frac{x(x+b)(x-b)}{(x+b)(x+b)}$, to its simplest form. *Ans.* $\frac{x^2-bx}{x+b}$.

5. Reduce $\frac{m^2-n^2}{m^2+2mn+n^2}=\frac{(m+n)(m-n)}{(m+n)(m+n)}$, to its simplest form. *Ans.* $\frac{m-n}{m+n}$.

6. Reduce $\frac{ax+bx}{ayz+byz}=\frac{(a+b)x}{(a+b)yz}$, to its simplest form. *Ans.* $\frac{x}{yz}$.

CASE II.

(43.) To reduce a mixed quantity to the form of a fraction.

RULE.

Multiply the integral part by the denominator of the fraction, to which product add the numerator, and under the result place the given denominator.

EXAMPLES.

1. Reduce $11x+\frac{x+y}{7x}$ to the form of a fraction.

In this example, the integral part is $11x$, which multiplied by the denominator $7x$, gives $77x^2$, to which, adding the numerator $x+y$, we have $77x^2+x+y$ for the numerator of the fraction sought, under which, placing the denominator $7x$, we finally obtain

$$\frac{77x^2+x+y}{7x}, \text{ for the reduced form of } 11x+\frac{x+y}{7x}.$$

2. Reduce $x-\frac{bx+x^2}{m}$ to the form of a fraction.

Ans. $\frac{mx-bx-x^2}{m}$.

3. Reduce $y+3x-\frac{6}{3+a}$ to the form of a fraction.

Ans. $\frac{3y+9x+ay+3ax-6}{3+a}$.

4. Reduce $x-\frac{a^2-b^2}{x}$ to the form of a fraction.

Ans. $\frac{x^2-a^2+b^2}{x}$.

5. Reduce $3a^2-6+\frac{6-x^2}{7-y}$ to the form of a fraction.

Ans. $\frac{21a^2-36-3a^2y+6y-x^2}{7-y}$.

6. Reduce $9+\frac{3x^2-8c^4}{a-x^2}$ to the form of a fraction.

Ans. $\frac{9a-6x^2-8c^4}{a-x^2}$.

7. Reduce $a+b+\frac{4}{a-b}$ to the form of a fraction.

Ans. $\frac{a^2-b^2+4}{a-b}$

8. Reduce $\frac{3m}{n+p}-x$ to the form of a fraction.

Ans. $\frac{3m-nx-px}{n+p}$.

9. Reduce $\frac{4+7a}{10}-5$ to the form of a fraction.

Ans. $\frac{7a-46}{10}$.

10. Reduce $\frac{4x}{3y+z}-8$ to the form of a fraction.

Ans. $\frac{4x-24y-8z}{3y+z}$.

11. Reduce $\frac{1-x}{1+x}+x-1$ to the form of a fraction.

Ans. $\frac{x^2-x}{x+1}$.

12. Reduce $\frac{4p-3q}{5}+p$ to the form of a fraction.

Ans. $\frac{9p-3q}{5}$.

CASE III.

(44.) To reduce a fraction to an integral, or mixed quantity.

RULE.

Divide the numerator by the denominator; the quotient will be the integral part: if there is a remainder, place it over the denominator for the fractional part.

EXAMPLES.

1. Reduce $\frac{9a-6x^2-8c^4}{a-x^2}$ to a mixed quantity.

Dividing the numerator by the denominator, we find this

FIRST OPERATION.

$$\begin{array}{l|l} 9a-6x^2-8c^4 & a-x^2 \\ \cline{2-2} 9a-9x^2 & 9=\text{integral part.} \\ \hline \qquad 3x^2-8c^4 = \text{numerator of fractional part.} & \end{array}$$

Therefore the quantity sought is $9+\frac{3x^2-8c^4}{a-x^2}$.

We will now change the order of the terms of the numerator and denominator by placing the x^2 first; we thus find this

SECOND OPERATION.

$$\begin{array}{l|l} -6x^2+9a-8c^4 & -x^2+a \\ \cline{2-2} -6x^2+6a & \quad 6=\text{integral part.} \\ \hline \qquad 3a-8c^4 = \text{numerator of fractional part.} & \end{array}$$

Therefore, the quantity sought is $6+\frac{3a-8c^4}{a-x^2}$.

These two results are equivalent, but under different forms.

2. Reduce $\frac{ax-x^2}{x}$ to an integral quantity. *Ans.* $a-x$.

3. Reduce $\frac{6x^2-ax}{3x+1}$ to a mixed quantity. *Ans.* $2x-\frac{2x+ax}{3x+1}$.

4. Reduce $\frac{m^3-y^3}{m-y}$ to an integral quantity.
Ans. m^2+my+y^2.

5. Reduce $\frac{20a^2-10a+6}{5a}$ to a mixed quantity.
Ans. $4a-2+\frac{6}{5a}$.

6. Reduce $\frac{9y^3-18y+8a^2y^2}{9y}$ to a mixed quantity.
Ans. $y^2-2+\frac{8a^2y}{9}$.

7. Reduce $\frac{14m^3-21n}{7m}$ to a mixed quantity

Ans $2m^2-\frac{3n}{m}$.

8. Reduce $\frac{12x^3-6x+4}{6x}$ to a mixed quantity

Ans. $2x^2-1+\frac{2}{3x}$

9. Reduce $\frac{a^3-3a^2b+3ab^2-b^3}{a^2-2ab+b^2}$ to an integral quantity.

Ans. $a-b$.

10. Reduce $\frac{a^2-2ab+b^2}{a-b}$ to an integral quantity. *Ans.* $a-b$.

11. Reduce $\frac{14m^2-7m+1}{7}$ to a mixed quantity.

Ans. $2m^2-m+\frac{1}{7}$.

12. Reduce $\frac{6x^2-3y^2+z^2}{3}$ to a mixed quantity.

Ans. $2x^2-y^2+\frac{z^2}{3}$.

CASE IV.

(45.) To reduce fractions to a common denominator.

RULE.

Multiply successively each numerator into all the denominators, except its own, for new numerators, and all the denominators together for a common denominator.

EXAMPLES.

1. Reduce $\frac{a}{x}$, $\frac{b}{2}$, $\frac{c}{7a}$, to equivalent fractions having a common denominator.

$a\times2\times7a=14a^2$ =new numerator of first fraction.
$b\times x\times7a=7abx$ =new numerator of second fraction.
$c\times x\times2\ =2cx$ =new numerator of third fraction.
and $x\times2\times7a=14ax$=common denominator.

Therefore, $\frac{14a^2}{14ax}$; $\frac{7abx}{14ax}$; $\frac{2cx}{14ax}$; are the equivalent fractions sought.

2. Reduce $\frac{3m}{2a}$, $\frac{2b}{3x}$, and y, to fractions having a common denominator. *Ans.* $\frac{9mx}{6ax}$; $\frac{4ab}{6ax}$; $\frac{6axy}{6ax}$.

3. Reduce $\frac{1}{2}$, $\frac{x^2}{3}$, $\frac{a^2+x^2}{a+x}$, to equivalent fractions having a common denominator. *Ans.* $\frac{3a+3x}{6a+6x}$; $\frac{2ax^2+2x^3}{6a+6x}$; $\frac{6a^2+6x^2}{6a+6x}$.

4. Reduce $\frac{x}{3b}$, $\frac{6x^2}{5c}$, $\frac{a^2-x^2}{d}$, to fractions having a common denominator. *Ans.* $\frac{5cdx}{15bcd}$; $\frac{18bdx^2}{15bcd}$; $\frac{15a^2bc-15bcx^2}{15bcd}$.

5. Reduce $\frac{a}{2}$, $\frac{b}{3}$, $\frac{c}{4}$, and $\frac{d}{6}$, to fractions having a common denominator. *Ans.* $\frac{72a}{144}$; $\frac{48b}{144}$; $\frac{36c}{144}$; $\frac{24d}{144}$.

6. Reduce $\frac{a}{2b}$, $\frac{b}{4c}$, $\frac{c}{6d}$, to fractions having a common denominator. *Ans.* $\frac{24acd}{48bcd}$; $\frac{12b^2d}{48bcd}$; $\frac{8bc^2}{48bcd}$.

7. Reduce $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, to fractions having a common denominator.

Ans. $\frac{bc}{abc}$; $\frac{ac}{abc}$; $\frac{ab}{abc}$.

8. Reduce $\frac{3a}{4}$, $\frac{5a}{3}$, $\frac{7a}{5}$, to fractions having a common denominator. *Ans.* $\frac{45a}{60}$; $\frac{100a}{60}$; $\frac{84a}{60}$

9. Reduce $\frac{6+a}{7}$, $\frac{5+b}{6}$, $\frac{c}{2}$, to fractions having a common denominator. *Ans.* $\frac{72+12a}{84}$; $\frac{70+14b}{84}$; $\frac{42c}{84}$

10. Reduce $\frac{6}{x+1}$, $\frac{4}{x-1}$, a, to fractions having a common denominator. *Ans.* $\frac{6x-6}{x^2-1}$; $\frac{4x+4}{x^2-1}$; $\frac{ax^2-a}{x^2-1}$.

CASE V.

(46.) To add fractional quantities.

RULE.

Reduce the fractions to a common denominator; then add the numerators, and place their sum over the common denominator.

EXAMPLES.

1. What is the sum of $\frac{x}{3a}$, $\frac{1}{3}$, $\frac{y}{7}$?

These fractions, when reduced to a common denominator, become $\frac{21x}{63a}$, $\frac{21a}{63a}$, $\frac{9ay}{63a}$; adding their numerators, we have $21x+21a+9ay$; placing this over the common denominator, we find

$$\frac{x}{3a}+\frac{1}{3}+\frac{y}{7}=\frac{21x+21a+9ay}{63a}=\frac{7x+7a+3ay}{21a}.$$

2. What is the sum of $3x+\frac{2x}{5}$ and $x-\frac{8x}{9}$? *Ans.* $3x+\frac{23x}{45}$.

3. What is the sum of $\frac{2x}{3}$, $\frac{7x}{4}$, $\frac{2x+1}{5}$? *Ans.* $2x+\frac{49x+12}{60}$

4. What is the sum of $\frac{3x}{4}$, $\frac{4x}{5}$, $\frac{5x}{6}$?

Ans. $\frac{45x+48x+50x}{60}=2x+\frac{23x}{60}$.

5. What is the sum of $\frac{a+b}{2}$, $\frac{a-b}{2}$? *Ans.* a.

6. What is the sum of $\frac{a^2+2ab+b^2}{4}$, $\frac{a^2-2ab+b^2}{4}$?

Ans. $\frac{a^2+b^2}{2}$.

7. What is the sum of $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$? *Ans.* $\frac{bc+ac+ab}{abc}$.

8. What is the sum of $\frac{x}{2}$, $\frac{x}{3}$, $\frac{x}{4}$, $\frac{x}{6}$? *Ans.* $x+\frac{x}{4}$.

9. What is the sum of $\frac{a}{2}$, $\frac{b}{3}$, $\frac{c}{4}$? *Ans.* $\frac{6a+4b+3c}{12}$.

10. What is the sum of $\frac{a}{a+b}$, $\frac{b}{a-b}$? *Ans.* $\frac{a^2+b^2}{a^2-b^2}$.

11. What is the sum of $\frac{1}{x+y}$, $\frac{1}{x-y}$? *Ans.* $\frac{2x}{x^2-y^2}$.

CASE VI.

(47.) To subtract one fraction from another.

RULE

Reduce the fractions to a common denominator; then subtract the numerator of the subtrahend from the numerator of the minuend, and place the difference over the common denominator.

EXAMPLES.

1. From $\frac{3x+a}{4}$ subtract $\frac{2x-a}{3}$.

These fractions, when reduced to a common denominator, become $\frac{9x+3a}{12}$ and $\frac{8x-4a}{12}$. Subtracting the numerators, we have $9x+3a-(8x-4a)=x+7a$; placing this over the common denominator 12, we find

$$\frac{3x+a}{4}-\frac{2x-a}{3}=\frac{x+7a}{12}.$$

2. From $\frac{6m+y}{5}$ subtract $\frac{m+y}{4}$. *Ans.* $\frac{19m-y}{20}$.

3. From $3y+\frac{y}{a}$ subtract $y-\frac{y-a}{c}$. *Ans.* $2y+\frac{cy+ay-a^2}{ac}$.

4. From $\frac{x+y}{2}$ subtract $\frac{x-y}{2}$. *Ans.* y.

5. From $\frac{x^2+2xy+y^2}{4xy}$ subtract $\frac{x^2-2xy+y^2}{4xy}$. *Ans.* 1.

6. From $\frac{6-x}{2}$ subtract $a+\frac{3+y}{3}$. *Ans.* $2-a-\frac{3x+2y}{6}$.

7. From $\frac{1}{x-y}$ subtract $\frac{1}{x+y}$. *Ans.* $\frac{2y}{x^2-y^2}$.

8. From $\frac{1}{a}$ subtract $\frac{1}{b}$. *Ans.* $\frac{b-a}{ab}$.

9. From $\frac{a+b}{2}$ subtract $\frac{a-b}{2}$. *Ans.* b.

CASE VII.

(48.) To multiply fractional quantities together.

RULE.

If any of the quantities to be multiplied are mixed, they must, by Case II., be reduced to a fractional form; then multiply together all the numerators for a numerator, and all the denominators together for a denominator.

EXAMPLES.

1. Multiply $\frac{x+a}{2}$ by $\frac{x+b}{3}$.

The product of the numerators will be

$$(x+a)\times(x+b)=x^2+ax+bx+ab;$$

and the product of the denominators is $2\times3=6$.

Hence, $\frac{x+a}{2}\times\frac{x+b}{3}=\frac{x^2+ax+bx+ab}{6}$.

2. Multiply $\frac{x^2-b^2}{bc}$ by $\frac{x^2+b^2}{b+c}$. *Ans.* $\frac{x^4-b^4}{b^2c+bc^2}$.

3. What is the continued product of $\frac{3-x}{7}$, $\frac{4+x}{2}$, and $\frac{3}{7}$? *Ans.* $\frac{36-3x-3x^2}{98}$.

4. What is the product of $\frac{a+b}{2}$ by $\frac{a-b}{2}$? *Ans.* $\frac{a^2-b^2}{4}$.

5. What is the continued product of $\frac{2x}{m}$, $\frac{h-d}{x}$, $\frac{b}{c}$, and $\frac{1}{r-1}$? *Ans.* $\frac{2bhx-2bdx}{cmrx-cmx}$.

6. What is the product of $y+\frac{y+1}{b}$ by $\frac{y-1}{2b}$? *Ans.* $\frac{by^2-by+y^2-1}{2b^2}$.

7. What is the product of $\frac{1}{x+y}$ and $\frac{1}{x-y}$? *Ans.* $\frac{1}{x^2-y^2}$.

8. What is the product of $3+\frac{x}{2}$ and $\frac{x}{y}$? *Ans.* $\frac{6x+x^2}{2y}$.

9. What is the product of $\frac{x-y}{2}$, $\frac{2}{x^2-y^2}$, and $x+y$? *Ans.* 1.

CASE VIII.

(49.) To divide one fraction by another.

RULE.

If there are any mixed quantities, reduce them to a fractional form, by Case II. Then invert the divisor, and multiply as in Case VII.

EXAMPLES.

1. Divide $\frac{3x+7}{4}$ by $\frac{4x-1}{5}$.

If we invert the divisor, and then multiply, we have

$$\frac{3x+7}{4}\times\frac{5}{4x-1}=\frac{15x+35}{16x-4} \text{ for the quotient.}$$

2. Divide $\frac{x^2-y^2}{x}$ by $\frac{x^2+y^2}{y}$. *Ans.* $\frac{x^2y-y^3}{x^3+xy^2}$.

3. Divide $\frac{7a^2y^{-3}}{3m^3}$ by $\frac{4ay^2}{5m}$. *Ans.* $\frac{35a}{12m^2y^5}$.

4. Divide $\frac{y-b}{8cd}$ by $\frac{3cy}{4d}$. *Ans.* $\frac{y-b}{6c^2y}$.

5. What is the quotient of $\frac{x}{x-1}$ dvided by $\frac{x}{2}$? *Ans.* $\frac{2}{x-1}$.

6. Divide $\frac{x^2-y^2}{2}$ by $\frac{x-y}{4}$. *Ans.* $2x+2y$.

7. Divide $\frac{a}{3}$ by $\frac{b}{4}$. *Ans.* $\frac{4a}{3b}$.

8. Divide $\frac{x^2-2xy+y^2}{a}$ by $\frac{x-y}{b}$. *Ans.* $\frac{bx-by}{a}$.

9. Divide $\frac{3+x}{7}$ by $\frac{4-y}{5}$. *Ans.* $\frac{15+5x}{28-7y}$.

10. Divide $\frac{7-y}{x}$ by $\frac{4+y}{y}$. *Ans.* $\frac{7y-y^2}{4x+xy}$.

CHAPTER III.

SIMPLE EQUATIONS.

(50.) AN EQUATION is an expression of two equal quantities with the sign of equality placed between them.

The terms or quantities on the left-hand side of the sign of equality constitute the *first member* of the equation, those on the right constitute the *second member*. Thus,

$$x+2=a, \qquad (1)$$

$$\frac{x}{2}-1=b, \qquad (2)$$

$$3x+7=c, \qquad (3)$$

are equations. The first is read, "x increased by 2 equals a."

The second is read, "one-half of x diminished by 1 equals b."

The third is read, "three times x increased by 7 equals c."

(51.) Nearly all the operations of Algebra are performed by the aid of equations. The relations of a question or problem are first to be expressed by an equation containing known quantities as well as the unknown quantity. Afterwards we must make such transformations upon this equation as to bring the unknown quantity by itself on one side of the equation, by which means it becomes known.

(52.) *An equation of the first degree, or a simple equation*, is one, in which the unknown quantity has no power above the first degree.

(53.) *A quadratic equation* is an equation of the second degree; that is, the unknown quantity is involved to the second power, and to no greater power.

(54.) An equation of the third, fourth, &c., degrees, is one which contains the unknown quantity to the third, fourth, &c., degrees; but to no superior degree.

(55.) The following axioms will enable us to make many transformations upon the terms of an equation without destroying their equality.

AXIOMS.

I. If equal quantities be added to both members of an equation, the equality of the members will not be destroyed.

II. If equal quantities be subtracted from both members of an equation, the equality of the members will not be destroyed.

III. If both members of an equation be multiplied by the same quantity, the equality will not be destroyed.

IV. If both members of an equation be divided by the same quantity, the equality will not be destroyed.

CLEARING EQUATIONS OF FRACTIONS.

(56.) When some of the terms of an equation are fractional, it is necessary to so transform it as to cause the denominators to disappear, which process is called *clearing of fractions.*

Let it be required to clear of fractions the following equation:

$$1+\frac{x}{2}+\frac{x}{3}-\frac{x}{6}=x. \qquad (1)$$

Now, by Axiom III., we can multiply all the terms of this equation by any number we please, without destroying the equality. If we multiply by a multiple of all the denominators, it is evident they will disappear.

If we choose the least multiple of the denominators as a multiplier, it is plain that the labor of multiplying will be the least possible.

Thus, in the above example, multiplying all the terms of both sides of the equation by 6, which is the least multiple of 2, 3, and 6, we have

$$6+3x+2x-x=6x. \qquad (2)$$

This equation is now free of fractions.

(57.) Hence, to clear an equation of fractions, we deduce, from what has been said, this

RULE.

Multiply all the terms of the equation by any multiple of their denominators. If we choose the least common multiple of the denominators for our multiplier, the terms, when cleared of fractions, will be in their simplest form.

EXAMPLES.

1. Clear of fractions, the equation $\frac{x-a}{5}=\frac{x+b}{2}-\frac{1}{7}$.

In this example, the least common multiple of the denominators 5, 2, and 7 is 70. Hence, multiplying all the terms of our equation by 70, we find

$$14x-14a=35x+35b-10,$$

for the equation when cleared of fractions.

2. Clear of fractions $\frac{x-2}{8}+\frac{x+a}{4}-\frac{x-b}{2}=x+\frac{x}{16}$.

Our multiplier will, in this example, be 16. We proceed by saying, 16 times one-eighth of any quantity is twice that quantity; hence, 16 times the first fraction is the same as twice the numerator, that is, $2x-4$. Again, 16 times one-fourth of any quantity is 4 times that quantity; hence, 16 times the second fraction is the same as 4 times the numerator, that is, $4x+4a$. In the same way 16 times the third fraction is the same as 8 times its numerator; and since this fraction has the negative sign, it is required to be subtracted, which is effected by changing the signs of the terms constituting the numerator (Art. 20). Hence, this third fraction, when cleared of its denominator, and the signs properly changed, gives $-8x+8b$. Multiplying each term of the second member of the equation by 16, we have $16x+x$. The equation when thus cleared becomes

$$2x-4+4x+4a-8x+8b=16x+x.$$

(58.) We must observe that when a fraction has the sign $-$, its value is required to be subtracted, so that, if it is written without the denominator, when cleared of fractions, all the signs of its numerator must be changed.

3. Clear of fractions $\frac{x-1}{2}+\frac{x+1}{3}-\frac{x-3}{4}=a+b-\frac{c}{7}$.

Ans. $42x-42+28x+28-21x+63=84a+84b-12c.$

4. Clear the equation, $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}+\frac{x}{6}=251$, of fractions.

Ans. $30x+20x+15x+12x+10x=15060.$

5. Clear the equation, $\frac{a^2}{x}+\frac{b}{m}+\frac{c}{d}=g$, of fractions.

Ans. $a^2dm+bdx+cmx=dgmx.$

6. Clear the equation, $\frac{x-1}{3}-\frac{x-a}{9}=\frac{x+b}{6}-4$, of fractions.

Ans. $6x-6-2x+2a=3x+3b-72.$

7. Clear the equation, $\frac{x-a}{b}+\frac{x-b}{c}=\frac{x-c}{d}$, of fractions.

Ans. $cdx-acd+bdx-b^2d=bcx-bc^2.$

8. Clear the equation, $\frac{x}{2}+\frac{a}{3}+\frac{b}{4}=\frac{c}{6}$, of fractions.

Ans. $6x+4a+3b=2c.$

9. Clear the equation, $\frac{x-1}{2}+\frac{x-2}{3}+\frac{x-3}{4}=\frac{x}{5}$, of fractions.

Ans. $30x-30+20x-40+15x-45=12x.$

10. Clear the equation, $\frac{a}{x}+\frac{3}{4}=\frac{5}{6}$, of fractions.

Ans. $12a+9x=10x$.

11. Clear the equation, $\frac{a}{b}+\frac{c}{x}+\frac{d}{x}=\frac{m}{x}$, of fractions.

Ans. $ax+bc+bd=bm$.

TRANSPOSITION OF THE TERMS OF AN EQUATION.

(59.) The next thing to be attended to, after clearing the equation of fractions, is to transform it so that all the terms containing the unknown quantity may constitute one member of the equation.

If we take the equation

$$\frac{a}{x}-\frac{1}{2}+\frac{b}{3}=8, \qquad (1)$$

we have, when cleared of fractions,

$$6a-3x+2bx=48x. \qquad (2)$$

If we add to both members of this equation $3x-2bx$ (Axiom I.), it becomes

$$6a-3x+2bx+3x-2bx=48x+3x-2bx. \qquad (3)$$

All the terms of the left-hand member cancel each other, except $6a$.

Therefore, we have

$$6a=48x+3x-2bx, \qquad (4)$$

in which all the terms of the right-hand member contain x.

If we compare equation (4) with (2), we shall discover that the terms $-3x+2bx$, which are on the left side of equation (2), are on the right side of equation (4) with their signs changed.

Hence, we conclude that the terms of an equation may change sides, provided they change signs at the same time.

(60.) To transpose a term from one side of an equation to the other, we must observe this

RULE.

Any term may be transposed from one side of an equation to the other, by changing its sign.

EXAMPLES.

1. Clear the equation $\frac{x+6}{2}+26=\frac{5x}{4}+2$ of fractions, and transpose the terms so that all those containing x may constitute the right-hand member.

First, clearing the above equation of fractions, by Rule under Art. 57, we have

$$2x+12+104=5x+8.$$

Secondly, transposing $2x$ from the left member to the right member, and 8 from the right member to the left, we have

$$12+104-8=5x-2x,\text{ for the result required.}$$

2. Clear the equation $\frac{x}{2}-\frac{a+x}{3}=7\frac{1}{2}$ of fractions, and transpose the terms. *Ans.* $3x-2x=45+2a$.

3. Clear of fractions the equation $\frac{7x}{9}-3\frac{1}{3}+\frac{x}{6}=\frac{x}{9}+2$, and transpose the terms. *Ans.* $14x+3x-2x=36+60$.

4. Clear of fractions and transpose the terms of the equation, $\frac{x}{a-b}-\frac{2+x}{a+b}=\frac{c}{a^2-b^2}$. *Ans.* $ax+bx-ax+bx=c+2a-2b$.

(61.) We are now prepared to find the value of the unknown quantity. If we take the answer to the last example, it may be written thus,

$$(a+b-a+b)\,x=c+2a-2b.$$

Uniting the like terms within the parenthesis, it becomes

$$2bx=c+2a-2b.$$

Dividing both sides of this equation by $2b$ (Axiom IV.), we find

$$x=\frac{c+2a-2b}{2b}.$$

Hence, the value of x is now known, since it is equal to the expression

$$\frac{c+2a-2b}{2b}.$$

(62.) It is obvious that the terms of an equation may be so transposed as to cause all the terms to constitute but one member of the equation, and then the other member will be zero.

Thus, if $\qquad x+6-a=3-2x,$

then will $\qquad x+2x+6-3-a=0.$

If we so transpose the terms of this last equation, as to make them constitute the right-hand member, we shall have

$$0=-x-2x-6+3+a.$$

In the higher parts of Analysis, equations are frequently reduced to this form, so that the positive and negative quantities cancel each other.

We may change the signs of all the terms of both members of an equation, without transposing them. Thus, the equation

$$c-3x-2a+x+b=m-4x+a,$$

with signs changed, becomes

$$-c+3x+2a-x-b=-m+4x-a.$$

This result would have been obtained by multiplying each term by -1 (Axiom III.).

(63.) From what has been done, we discover that an equation of the first degree may be resolved by the following general

RULE.

I. If any of the terms of the equation are fractional, the equation must be cleared of fractions, by Rule under Art. 57.

II. The terms must then be so transposed that all those containing the unknown quantity may constitute one side or member of the equation, by Rule under Art. 60.

III. Then divide the algebraic sum of those terms on that side of the equation which are independent of the unknown quantity, by the algebraic sum of the coefficients of the terms containing the unknown quantity, and the quotient will be the value of the unknown quantity.

EXAMPLES.

1. In the equation $\frac{x}{3}+\frac{x}{4}=x-10$, what is the value of x?

This cleared of fractions becomes

$$4x+3x=12x-120.$$

When the terms are transposed and united, we have

$$120=5x.$$

Dividing by 5, we get

$$24=x.$$

2. What is the value of x in $x-\frac{2x+1}{3}=\frac{x+3}{4}$?

Ans. $x=13$.

3. Given $\frac{21-3x}{3}-\frac{4x+6}{9}=6-\frac{5x+1}{4}$, to find x.

Ans. $x=3$.

4. Find x from the equation $3ax+\frac{a}{2}-3=bx-a$.

Ans. $x=\frac{6-3a}{6a-2b}$.

5. Given $\frac{x-2}{4}-\frac{3x}{2}+\frac{15x}{2}=37$, to find x. *Ans.* $x=6$.

6. Given $\frac{3cx}{a}-\frac{2bx}{m}-4=f$, to find x. *Ans.* $x=\frac{afm+4am}{3cm-2ab}$.

7. Find x from the equation $\frac{8nx-b}{7}-\frac{3b}{2}=4-b-\frac{c}{2}$.

Ans. $x=\frac{56+9b-7c}{16n}$.

8. Given $\frac{x-1}{2}+\frac{x+1}{2}=3x-12$, to find x. *Ans.* $x=6$.

9. Given $x-\frac{3x-5}{13}+\frac{4x-2}{11}=x-1$, to find x. *Ans.* $x=6$.

10. Given $\frac{x+7}{3}-3x+\frac{6x-2}{5}+3=x$, to find x. *Ans.* $x=2$.

11. Given $\frac{3x-2}{7}+\frac{3x+2}{11}=x-1$, to find x. *Ans.* $x=3$.

12. Given $\frac{x}{3}-\frac{x}{7}+x=11$, to find x. *Ans.* $x=9\frac{6}{25}$.

13. Given $\frac{x}{2}-\frac{x+2}{3}=\frac{x}{18}$, to find x. *Ans.* $x=6$.

The true value of the unknown quantity, substituted for its representative, will cause the two members of the given equation to become identical.

In the last example, if we write 6 for x, the equation will become

$\frac{6}{2}-\frac{6+2}{3}=\frac{6}{18}$, or $3-2\frac{2}{3}=\frac{1}{3}$, where each member becomes $\frac{1}{3}$.

This method of verifying the solution of simple equations, should be used until the student becomes perfectly familiar with it.

14. Given $\frac{x-1}{4}+\frac{x+5}{10}=3-\frac{x}{5}$, to find x. *Ans.* $x=5$

This value of x causes each member of the equation to become 2.

15. Given $\frac{x-3}{4}-\frac{11-x}{2}=\frac{x+5}{2}-7$, to find x. *Ans.* $x=7$.

This value of x causes each member of the equation to become -1.

16. Given $\frac{x-3}{5}+\frac{x-5}{7}=\frac{x-2}{35}$, to find x. *Ans.* $x=4$.

This value of x causes each side of the equation to become $\frac{2}{35}$.

17. Given $\frac{4}{x+1}=\frac{6}{x+2}$, to find x. *Ans.* $x=1$.

This value of x causes each member of the equation to become 2.

18. Given $\frac{x-a}{2}-\frac{x-b}{3}=\frac{x-c}{4}$, to find x. *Ans.* $x=4b+3c-6a$.

19. Given $\frac{x-1}{2}+\frac{x-2}{3}-\frac{x-3}{4}=x-5$, to find x. *Ans.* $x=11$.

20. Given $\frac{2x-3}{7}-\frac{4x+2}{5}=x-19$, to find x. *Ans.* $x=12$.

21. Given $\frac{4+x}{5}-\frac{5+x}{6}=\frac{6+x}{7}-1$, to find x. *Ans.* $x=1$.

22. Given $\frac{2x-4}{x}=\frac{1}{x}+\frac{3}{4}$, to find x. *Ans.* $x=4$.

23. Given $\frac{4x-1}{5}-\frac{x-3}{4}=\frac{5x+2}{8}$, to find x. *Ans.* $x=4$.

24. Given $\frac{x-1}{2}+\frac{x-1}{3}+\frac{x-1}{4}=13$, to find x.

Ans. $x=13$.

25. Given $\frac{x+a}{2}+\frac{x+b}{4}=\frac{x+c}{3}$, to find x.

Ans. $x=\frac{4c-6a-3b}{5}$.

26. Given $\frac{x+1}{a}+\frac{x+2}{b}=c$, to find x.

Ans. $x=\frac{abc-2a-b}{a+b}$.

27. Given $\frac{x-a}{2}+\frac{x+a}{3}=\frac{x}{4}$, to find x. *Ans.* $x=\frac{2a}{7}$.

QUESTIONS, THE SOLUTIONS OF WHICH REQUIRE EQUATIONS OF THE FIRST DEGREE.

(64.) In the solution of questions, by the aid of Algebra, the most difficult part is to obtain the proper equation which shall include all the necessary relations of the question. When once this equation of condition is properly found, the value of the unknown quantity is readily obtained by the Rule under Art. 63.

Suppose we wish to solve, by Algebra, the following question:

1. What number is that, whose half increased by its third part and one more, shall equal itself?

If we suppose x to be the number sought, its half will be $\frac{x}{2}$, which, increased by its third part, becomes $\frac{x}{2}+\frac{x}{3}$, and this increased by one, becomes $\frac{x}{2}+\frac{x}{3}+1$, which by the question must equal the number.

Therefore, we have $\frac{x}{2}+\frac{x}{3}+1=x$ for the equation of condition.

Solving this by Rule under Art. 63, we have $x=6$.

VERIFICATION.

$$\frac{x}{2}=\tfrac{1}{2}\text{ of }6=3$$

$$\frac{x}{3}=\tfrac{1}{3}\text{ of }6=2$$

$$1\qquad\quad=1$$

Therefore, $\frac{x}{2}+\frac{x}{3}+1=6$, which shows that 6 is truly the number sought.

Again, let us endeavor to solve this question:

2. What number is that whose third part exceeds its fourth part by 5?

Suppose x to be the number, then will its third part $=\frac{x}{3}$; its fourth part $=\frac{x}{4}$.

Therefore, the excess of its third part over its fourth part is expressed by $\frac{x}{3}-\frac{x}{4}$, which, by the question, must equal 5.

Hence, we have the following equation, $\frac{x}{3}-\frac{x}{4}=5$.

This solved, gives $x=60$; the third part of which is 20, and its fourth part is 15, so that its third part exceeds its fourth part by 5; hence, this is the number sought.

(65.) The method of forming an equation from the conditions of a question, is of such a nature as not to admit of any one simple rule, but must be in a measure left to the ingenuity of the student.

It will, however, be of assistance to pay attention to the following

RULE.

Having denoted the quantity sought by x, *or some other letter,* *we must indicate by algebraic symbols, the same operations as it would be necessary to perform upon the true number, in order to verify the conditions of the question.*

3. Out of a cask of wine which had leaked away a third part, 21 gallons were afterwards drawn, and the cask was then found to be half full. How much did it hold?

Suppose x to be the number of gallons which the cask held.

Then the part leaked away must be $\frac{x}{3}$.

And the part leaked away, together with the quantity drawn off, must be $\frac{x}{3}+21$.

Now, by the question, the cask is still half full, so that what has leaked out, together with what has been drawn off, must be $\frac{x}{2}$.

Hence, we have this equation, $\frac{x}{2}=\frac{x}{3}+21$,

which, cleared of fractions, becomes $3x=2x+126$.

Transposing and uniting terms, we have $x=126$.

4. There are two numbers which are to each other as 6 to 5, and whose difference is 40. What are the numbers?

Suppose the numbers to be denoted by $6x$ and $5x$, which are obviously as 6 to 5 for all values of x. By the question, the difference of these numbers is 40. Therefore, we have $6x-5x=40$, that is, $x=40$.

Hence, $\left.\begin{array}{l}6x=6\times40=240\\5x=5\times40=200\end{array}\right\}$ the numbers sought.

5. A farmer had two flocks of sheep, each containing the same number. Having sold from one of these 39, and from the other 93, he finds twice as many remaining in the one as in the other. How many did each flock originally contain?

Suppose the number in each flock to be denoted by x.

Then the flock from which he sold 39 will have remaining $x-39$.

And the one from which he sold 93 will have remaining $x-93$

Hence, by the question, we have

$$2\times(x-93)=x-39, \text{ or } 2x-186=x-39.$$

Transposing and uniting terms, $x=147$.

6. Divide the number 36 into three such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, $\frac{1}{4}$ of the third, shall be respectively equal to each other.

If we denote the three parts by $2x$, $3x$, $4x$, it is plain that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, $\frac{1}{4}$ of the third, will be equal for all values of x.

Now by the question, the sum of these three parts must equal 36.

Therefore, $2x+3x+4x=36$.
Uniting terms, we have $9x=36$.
Dividing by 9, we obtain $x=4$.

Consequently, $\left.\begin{array}{l}2x=2\times4=8\\3x=3\times4=12\\4x=4\times4=16\end{array}\right\}$ the parts sought.

7. Two pieces of cloth are of the same price by the yard, but of different lengths: the one cost \$5, the other \$6½. If each piece had been 10 yards longer, their lengths would have been as 5 to 6. What was the length of each piece?

Since the price per yard was the same for both pieces, their lengths must have been to each other the same as the number of dollars which they cost, or as 5 to 6½, or, which is the same, as 10 to 13.

Therefore, we will denote their lengths, in yards, by $10x$ and $13x$.

These become, when increased by 10,

$$10x+10 \text{ and } 13x+10,$$

which, by the question, must be as 5 to 6.

Hence, $6\,(10x+10)=5\,(13x+10)$.

Expanding, $60x+60=65x+50$.

Transposing and uniting terms, we get $10=5x$, and $x=2$.

Therefore, $\left.\begin{aligned}10x=10\times2=20\\13x=13\times2=26\end{aligned}\right\}$ the lengths sought.

8. A person engaged a workman for 50 days. For each day that he labored he received 8 shillings, and for each day that he was idle he paid 5 shillings for his board. At the end of the 50 days the account was settled, when the laborer received 10 pounds 18 shillings. Required the number of working days, and the number of idle days.

Let x equal the number of days he worked. Then will $50-x$ equal the number of days he was idle.

Also, $8x$ equals amount, in shillings, which he earned, and $(50-x)\times5=250-5x$ equals amount of his board bill.

Therefore, he must receive $8x-(250-5x)=13x-250$, which, by the question, must equal £10 18*s.*=218*s.*

Hence, we have this condition,

$$13x-250=218.$$

This readily gives $x=36=$ number of days he worked, and $50-x=50-36=14=$ number of idle days.

9. Divide the number 237 into two such parts, that the one may be contained in the other $1\frac{1}{4}$ times. What are these parts?

If x equals the smaller part, then the larger part will be $1\frac{1}{4}x=\frac{5x}{4}$.

Adding both parts together, we get

$$x+\frac{5x}{4}=237.$$

This solved, gives $x=105\frac{1}{3}$, and consequently

$$\frac{5x}{4}=131\frac{2}{3}.$$

So that the parts are $105\frac{1}{3}$ and $131\frac{2}{3}$.

Had we put $4x=$ the smaller part, then the larger part would have been $5x$.

Now taking the sum of the parts, we get

$$4x+5x=237.$$

This solved, gives $x=26\frac{1}{3}$.

Consequently, $4x=105\frac{1}{3}$ and $5x=131\frac{2}{3}$.

10. Three persons, A., B., and C., draw prizes in a lottery. A. draws $200; B. draws as much as A., together with a third of what C. draws; and C. draws as much as A. and B. together. What is the amount of the prizes?

Let $x=$the number of dollars which C. drew.

A. drew $200, and B. drew $\$200+\frac{x}{3}$.

Since C.'s equals the sum of A.'s and B.'s, we have this condition:

$$x=200+200+\frac{x}{3}.$$

Consequently, $x=600=$C.'s money.

Therefore, A.'s = $200
B.'s = 400
C.'s = 600
Sum of the prizes = $1200

11. Two capitalists calculate their fortunes, and it appears that

one is twice as rich as the other, and that together they possess \$38700. What is the capital of each?

Ans. The one had \$12900, the other \$25800.

12. The sum of \$2500 is to be divided between two brothers, so that one receives \$4 as often as the other receives \$1. How much does each receive?

Ans. The one receives \$500, the other \$2000.

13. Two friends met a horse-dealer leading a horse, which they resolved to buy jointly. When they had agreed as to the price, they found that the one was able to pay only the fifth part of the price, and the other only the seventh part, which together amounted to \$48. What was the price of the horse?

Ans. \$140.

14. A fortress has a garrison of 2600 men; among whom there are nine times as many foot-soldiers, and three times as many artillery soldiers as cavalry. How many are there of each kind? *Ans.* 200 cavalry, 600 artillery, and 1800 foot.

15. Find a number, such, that being multiplied by 7, and the product increased by 3, and this sum divided by 2, and 4 being subtracted from the quotient, we shall obtain 15. *Ans.* 5.

16. One of my acquaintances is now 40 years old, his son is 9 years old. In how many years will this man, who is now more than 4 times as old as his son, be only twice as old as he is?

Ans. In 22 years.

17. A cook receives from his master \$$11\frac{1}{4}$, which he is to expend for the same number of chickens, ducks, and geese. Each chicken cost $\frac{1}{4}$ of a dollar, each duck $\frac{3}{8}$ of a dollar, and each goose $\frac{1}{2}$ of a dollar. How many did he buy of each kind?

Ans. 10.

18. A gentleman spent $\frac{1}{10}$ of his fortune in Holland, $\frac{1}{5}$ in France, $\frac{1}{8}$ in England, and $\frac{1}{4}$ in Italy. He has now but \$975 left. How much money had he at first? *Ans.* \$3000.

19. The sum of two numbers is fifty; but when the greater is multiplied by 6, and the smaller by 5, the sum of the two products is 276. What are the two numbers? *Ans.* 26 and 24.

20. A person was desirous of giving 3 cents apiece to some beggars, but found he had not money enough by 8 cents; he therefore gave them 2 cents apiece, and had 3 cents remaining. What was the number of beggars? *Ans.* 11 beggars.

21. Two persons, A. and B., lay out equal sums of money in trade: A. gains $126, and B. loses $87, and A.'s money is now double of B.'s. What did each lay out? *Ans.* $300.

22. The sum of $1200 is to be divided between two persons, A. and B., so that A.'s share shall be to B.'s share as 2 to 7. How much ought each to receive?

Ans. A. 266\frac{2}{3}$, B. 933\frac{1}{3}$.

23. Divide the number 46 into two unequal parts, so that the greater divided by 7, and the less by 3, the quotients together may amount to 10. What are these parts? *Ans.* 28 and 18.

24. In a company of 266 persons, consisting of officers, merchants, and students, there were four times as many merchants, and twice as many officers as students. How many were there of each class?

Ans. 38 students, 152 merchants, and 76 officers.

25. A field of 864 square rods is to be divided among three farmers, A., B., and C., so that A.'s part shall be to B.'s as 5 to 11, and C. may receive as much as A. and B. together. How much ought each to receive?

Ans. A. 135, B. 297, C. 432 square rods.

26. Divide $1520 among three persons, A., B., C., so that B. may receive $100 more than A.; and C. $270 more than B. How much ought each to receive?

Ans. A. $350, B. $450, C. $720.

27. A certain sum of money is to be divided among three persons, A., B., C., as follows: A. shall receive $3000 *less* than the half of it, B. $1000 *less* than the third part, and C. is to receive $800 *more* than the fourth part of the whole sum. What is the sum to be divided? and what does each receive?

Ans. The whole sum is $38400. A. receives $16200, B. $11800, C. $10400.

28. A mason, 12 journeymen, and 4 assistants receive together $72 wages for a certain time. The mason receives $1 daily, each journeyman $$\frac{1}{2}$, and each assistant $$\frac{1}{4}$. How many days must they have worked for this money? *Ans.* 9 days.

29. A servant receives from his master $40 wages, yearly, and a suit of livery. After he had served 5 months he asked for his discharge, and received for this time the livery, together with $$6\frac{1}{6}$ in money. How much did the livery cost? *Ans.* $18.

30. Find a number, such, that $\frac{1}{3}$ thereof increased by $\frac{1}{4}$ of the same, shall be equal to $\frac{1}{6}$ of it if increased by 35. *Ans.* 84.

31. A gentleman spends $\frac{2}{3}$ of his yearly income in board and lodging, $\frac{2}{3}$ of the remainder in clothes, and lays by £20 a year. What is his income? *Ans.* £180.

32. A gentleman bought a chaise, horse, and harness for $360. The horse cost twice as much as the harness, and the chaise cost twice as much as the harness and horse together. What was the price of each?

Ans. The harness cost $40.
The horse cost $80.
The chaise cost $240.

33. If a reservoir can be exhausted by one engine in 7 hours, by another in 8 hours, and by a third in 9 hours, in what time will it be exhausted, if they are all worked together?

Ans. In $2\frac{122}{191}$ hours.

34. In an orchard of fruit trees, $\frac{1}{2}$ of them bear apples, $\frac{1}{4}$ of

them pears, $\frac{1}{8}$ of them peaches, 7 trees bear cherries, 3 plums, and 2 quinces. How many trees are there in the orchard?

Ans. 96 trees.

(66.) It would be a very pleasant exercise, and at the same time a very useful one, for the student to give analytical solutions to all the foregoing questions of simple equations. As an illustration of the kind of analysis required, the student is referred to Chapter XII., Higher Arithmetic.

We here give, in full, the analysis of a few of the foregoing problems.

Problem 11.

Since the one is twice as rich as the other, the wealth which they together possess is three times as much as that possessed by the poorer one. But they together possessed \$38700; hence, one-third of this, \$12900, is what the poorer one possessed, and twice \$12900=\$25800, is what the other possessed.

Problem 12.

Since one receives \$4 as often as the other receives \$1, the one must receive $\frac{4}{5}$ of the whole, and the other $\frac{1}{5}$ of it. $\frac{4}{5}$ of \$2500=\$2000, $\frac{1}{5}$ of \$2500=\$500. Hence, \$2000 and \$500 are their respective shares.

Problem 13.

One-fifth and one-seventh, added, make $\frac{12}{35}$, which is \$48. If $\frac{12}{35}$ is \$48, $\frac{1}{35}$ is $\$48 \div 12 = \$4$; and the whole is $\$4 \times 35 = \$140$, the price of the horse.

Problem 14.

Since there are 9 times as many foot-soldiers, and 3 times as many artillery soldiers as cavalry, the whole number must be 13 times as many as the cavalry. But the whole number is 2600; hence, $2600 \div 13 = 200$, is the number of cavalry, $200 \times 9 = 1800$, is the number of foot-soldiers, and $200 \times 3 = 600$, is the number of artillery.

Problem 15.

If we commence with the result and reverse the successive operations, we shall evidently obtain the number sought. Thus, if 4 had not been subtracted, the result would have been $15+4=19$. If the division had not been performed, the result would have been $19 \times 2=38$. Again, if 3 had not been added, the result must have been $38-3=35$. This, divided by 7, gives 5 for the number sought.

Problem 16.

When the son was born the father must have been $40-9=31$ years of age. Hence, when the son reaches the age of 31, the father will be twice 31, that is, 62; but he is now 40, and in $62-40=22$ years more, he will be twice as old as his son.

Problem 17.

If he buys one of each kind, they will cost $\frac{1}{4}+\frac{3}{8}+\frac{1}{2}=\frac{9}{8}$ of a dollar. Now, he wishes to lay out $11\frac{1}{4}$ dollars, which is $(11\frac{1}{4} \div \frac{9}{8}=10)$ ten times $\frac{9}{8}$ of a dollar; consequently he must purchase 10 of each kind.

(67.) We will now proceed to show how problems may be generalized, that is, made to embrace all problems of a similar kind.

Problems 11 and 12, generalized, become as follows:

35. Find two numbers, such that the one may be m times as great as the other, and that their sum may equal a. What are these numbers?

If x represent the smaller number, then will mx represent the larger.

And $x+mx=a$, consequently $x=\frac{a}{1+m}$, and $mx=\frac{ma}{1+m}$.

So that the two numbers are $\frac{a}{1+m}$ and $\frac{ma}{1+m}$.

In order that these general results may agree with Prob. 11, we must have $a=\$38700$, $m=2$.

Using these values, we have

$$\left.\begin{aligned}\frac{a}{1+m}&=\frac{\$38700}{1+2}=\$12900.\\ \frac{ma}{1+m}&=\frac{2\times\$38700}{1+2}=\$25800.\end{aligned}\right\}\text{The results correspond with } \textit{Ans.} \text{ to Prob. 11.}$$

If $a=\$2500$, and $m=4$, we shall find

$$\frac{a}{1+m}=\frac{\$2500}{1+4}=\$500.$$

$$\frac{ma}{1+m}=\frac{4\times\$2500}{1+4}=\$2000.$$

These results correspond with the answers to Prob. 12.

Prob. 13 generalized, becomes as follows:

36. Find a number, such, that when it is divided by m and by n, and the quotients afterwards added, the sum shall be a. What is this number?

If x denote the number sought, we shall have this condition:

$$\frac{x}{m}+\frac{x}{n}=a.$$

This immediately gives $x=\frac{mna}{m+n}$.

If we take $a=\$48$, $m=5$, $n=7$, we shall have

$$\frac{mna}{m+n}=\frac{5\times7\times\$48}{5+7}=\$140.$$

which agrees with Prob. 13.

The generalization of Problems 14 and 24 may be made as follows:

37. Divide the number a into three parts, such that the second may be m times, and the third n times as great as the first. What are these parts?

Let x represent the first part.

Then will mx represent the second part.

And nx will represent the third part.

Now, since the three parts make a, we have this condition:

$$x+mx+nx=a.$$

This readily gives

$$x=\frac{a}{1+m+n}=\text{first part.}$$

$$mx=\frac{ma}{1+m+n}=\text{second " }$$

$$nx=\frac{na}{1+m+n}=\text{third " }$$

If we take $a=2600$ *men*, $m=3$, and $n=9$, we shall have

$$\frac{a}{1+m+n}=\frac{2600\ men}{1+3+9}=200\ men\text{, for cavalry.}$$

$$\frac{ma}{1+m+n}=\frac{3\times2600\ men}{1+3+9}=600\ men\text{, for artillery.}$$

$$\frac{na}{1+m+n}=\frac{9\times2600\ men}{1+3+9}=1800\ men\text{, for foot-soldiers.}$$

These results agree with answers to Problem 14.

If we take $a=266$ persons, $m=2$, and $n=4$, we shall find

$$\frac{a}{1+m+n}=\frac{266}{1+2+4}=38\text{ for number of students.}$$

$$\frac{ma}{1+m+n}=\frac{2\times266}{1+2+4}=76\text{ for number of officers.}$$

$$\frac{na}{1+m+n}=\frac{4\times266}{1+2+4}=152\text{ for number of merchants.}$$

These results agree with Problem 24.

Problem 23 may be generalized as follows:

38. Divide the number a into two such parts, that the sum of the quotients which are obtained, when the one part is divided by m and the other by n, may be equal to the number b. What are these parts?

If x represent one of the parts of a, the other part will be represented by $a-x$. And we shall have this condition:

$$\frac{x}{m}+\frac{a-x}{n}=b.$$

From this we immediately find

$$x=\frac{ma-mnb}{m-n}, \text{ first part.}$$

$$a-x=\frac{mnb-na}{m-n}, \text{ second part.}$$

If we take $a=46$, $m=7$, $n=3$, and $b=10$, we shall have

$$\frac{ma-mnb}{m-n}=\frac{7\times 46-7\times 3\times 10}{7-3}=28,$$

$$\frac{mnb-na}{m-n}=\frac{7\times 3\times 10-3\times 46}{7-3}=18.$$

These results agree with Problem 23.

(68.) We will add a few more Problems, giving both the particular and general form.

39. Divide 200 into two such parts, that the difference of their squares may be 400. What are the parts?

Ans. 99 and 101.

40. Divide a into two such parts, that the difference of their squares may be b. What are the parts?

Ans. $\frac{a^2-b}{2a}, \frac{a^2+b}{2a}$.

41. Find two numbers whose difference is 4, and the difference of whose squares is 80.

Ans. 12 and 8.

42. Find two numbers whose difference is a, and the difference of whose squares is b. *Ans.* $\frac{b+a^2}{2a}$ and $\frac{b-a^2}{2a}$.

43. A courier who had started from a certain place 10 days ago, is pursued by another from the same place, and by the same way. The first goes 4 miles every day, the other 9. How many days will be required for the second to overtake the first? *Ans.* 8 days.

44. A courier left this place n days ago, and makes a miles daily. He is pursued by another making b miles daily. How many days will be required for the second to overtake the first?

Ans. $\frac{na}{b-a}$.

EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

(69.) In this case, we must endeavor to deduce from the given equations, one single equation containing only one unknown quantity; that is, we must cause all the unknown quantities, except one, to disappear, which process is called *Elimination.*

FIRST METHOD.

ELIMINATION BY ADDITION AND SUBTRACTION.

1. Suppose we have given the two equations,

$$\left.\begin{array}{l} x+y=19, \\ x-y=11, \end{array}\right\}$$

Taking the sum of these two equations, we shall eliminate y, and find

$$2x=30,$$
$$\text{or,}\quad x=15.$$

Subtracting the second from the first, we shall eliminate x, and obtain

$$2y=8,$$
$$\text{or,}\quad y=4.$$

2. We have given the two equations

$$\frac{x}{3}+\frac{y}{4}=8, \qquad (1)$$

$$\frac{x}{6}+\frac{y}{16}=3, \qquad (2)$$

to find the value of x and y.

We will clear these equations of fractions, by multiplying the first by 12, and the second by 48; we thus obtain

$$4x+3y=96, \qquad (3)$$
$$8x+3y=144. \qquad (4)$$

Subtracting (3) from (4), we eliminate y, and obtain

$$4x=48. \qquad (5)$$

Dividing by 4, we find

$$x=12.$$

Subtracting (5) from (3), we shall eliminate x, and find

$$3y=48. \qquad (6)$$

Dividing by 3, we find

$$y=16.$$

3. If we have given the two equations

$$2x-3y=4, \qquad (1)$$
$$8x-6y=40, \qquad (2)$$

to find x and y, we proceed as follows:

Dividing (2) by 2, it becomes

$$4x-3y=20. \qquad (3)$$

Subtracting (1) from (3), we shall eliminate y, and obtain

$$2x=16. \qquad (4)$$

Consequently, $x=8.$

Subtracting (1) from (4), we shall eliminate x, and obtain

$$3y=12.$$

Consequently, $y=4.$

From what has been done, we discover that an unknown quantity may be eliminated from two equations by the following

RULE.

Operate upon the two given equations, by multiplication or division, so that the coefficients of the quantity to be eliminated, may become the same in both equations; then add the two equations, or subtract one from the other, as may be necessary, to cause these two terms to disappear.

EXAMPLES.

4. Given, to find x and y, the two equations,

$$3x - y = 3, \qquad (1)$$

$$y + 2x = 7. \qquad (2)$$

If we add the two equations, we have

$$5x = 10.$$

Therefore, $x = 2.$ (3)

Again, multiplying (3) by 2, we get

$$2x = 4. \qquad (4)$$

Subtracting (4) from (2), we obtain

$$y = 3.$$

5. Given, to find x and y, the two equations,

$$\frac{x}{2} + \frac{y}{3} = 6, \qquad (1)$$

$$\frac{x}{3} + \frac{y}{2} = 6\tfrac{1}{2}. \qquad (2)$$

Clearing these equations of fractions, by multiplying each by 6, they become

$$3x + 2y = 36, \qquad (3)$$

$$2x + 3y = 39. \qquad (4)$$

Multiplying (3) by 3, and (4) by 2, they become

$$9x+6y=108, \qquad (5)$$
$$4x+6y=\ 78. \qquad (6)$$

Subtracting (6) from (5), we get

$$5x=30,$$

and $$x=6. \qquad (7)$$

Multiplying (7) by 2, it becomes

$$2x=12. \qquad (8)$$

Subtracting (8) from (4), we find

$$3y=27.$$

Therefore, $$y=9.$$

6. Suppose we wish to find x, y, and z, from the three equations,

$$5x-6y+4z=15, \qquad (1)$$
$$7x+4y-3z=19, \qquad (2)$$
$$2x+\ y+6z=46. \qquad (3)$$

We will first eliminate y; for this purpose multiply (3) first by 4 and then by 6, and we shall obtain

$$8x+4y+24z=184, \qquad (4)$$
$$12x+6y+36z=276. \qquad (5)$$

Add (1) to (5), and subtract (2) from (4), and we have

$$17x+40z=291, \qquad (6)$$
$$x+27z=165. \qquad (7)$$

We have now the two equations (6) and (7), and but two unknown quantities, x and z.

Hence, when we have three equations containing three unknowns, we must eliminate one of the unknowns, so as to obtain two new equations, containing only two unknowns.

In the case of four unknowns, we could eliminate one of these, and thus obtain three new equations, containing three unknowns.

We will now continue our solution as follows:

Multiply (7) by 17, and it will become

$$17x+459z=2805. \qquad (8)$$

Subtracting (6) from (8) we obtain

$$419z=2514. \qquad (9)$$

Dividing (9) by 419, we find

$$z=6. \qquad (10)$$

Multiplying (10) by 27, we find

$$27z=162. \qquad (11)$$

Subtracting (11) from (7), we get

$$x=3. \qquad (12)$$

Multiplying (10) by 6, and (12) by 2, and then taking their sum, we find

$$6z+2x=42. \qquad (13)$$

Subtracting (13) from (3), we get

$$y=4.$$

(70.) We will now repeat the solution of this last question, adopting a simple and easy method of indicating the successive steps in the operations.

The method which we propose to make use of, is to indicate by algebraic signs, the same operations upon the respective *numbers* of the different equations, as we wish to have performed upon the equations themselves. Thus,

$(6)=(4)\times 3$ { shows, that equation (6) is obtained by multiplying equation (4) by 3.

$(10)=(7)+(1)$ { shows, that equation (10) is obtained by adding equations (7) and (1).

$(11)=(6)-(3)$ { shows, that equation (11) is obtained by subtracting equation (3) from (6).

$$(15)=(14)\div 3 \left\{ \begin{array}{l} \text{shows, that equation (15) is obtained by di-} \\ \quad \text{viding equation (14) by 3.} \end{array} \right.$$

And so on for other combinations.

This kind of notation will become familiar by a little practice. We will now resume our equations of last example.

$$\text{Given} \left\{ \begin{array}{ll} 5x-6y+4z=15, & (1) \\ 7x+4y-3z=19, & (2) \\ 2x+\ \ y+6z=46, & (3) \end{array} \right\} \text{to find } x, y, \text{ and } z.$$

$$\begin{array}{rl}
8x+\ 4y+\ 24z=\ 184, & (4)=\ (3)\times 4 \\
12x+\ 6y+\ 36z=\ 276, & (5)=\ (3)\times 6 \\
17x+\ 40z=\ 291, & (6)=\ (1)+(5) \\
x+\ 27z=\ 165, & (7)=\ (4)-(2) \\
17x+459z=2805, & (8)=\ (7)\times 17 \\
419z=2514, & (9)=\ (8)-(6) \\
z=6, & (10)=\ (9)\div 419 \\
27z=162, & (11)=(10)\times 27 \\
x=3, & (12)=\ (7)-(11) \\
6z=36, & (13)=(10)\times 6 \\
2x=6, & (14)=(12)\times 2 \\
6z+2x=42, & (15)=(13)+(14) \\
y=4. & (16)=\ (3)-(15)
\end{array}$$

Collecting equations (12), (16), and (10), we have

$$\textit{Ans.} \left\{ \begin{array}{ll} x=3, & (12) \\ y=4, & (16) \\ z=6. & (10) \end{array} \right.$$

The foregoing method of indicating the successive steps of algebraic operations will be found convenient in blackboard work in class exercises. It presents at once, the precise steps employed by the pupil, and obviates the necessity of inquiring by what process any particular result has been obtained.

SECOND METHOD.

(71.) ELIMINATION BY SUBSTITUTION.

1. Suppose we have given the two equations,

$$5x+2y=45, \qquad (1)$$
$$4x+\ y=33. \qquad (2)$$

From the first we find

$$y=\frac{45-5x}{2}. \qquad (3)$$

Substituting this value of y in (2), we have

$$4x+\frac{45-5x}{2}=33. \qquad (4)$$

Equation (4), when cleared of fractions, becomes

$$8x+45-5x=66. \qquad (5)$$

This gives $x=7.$

Substituting this value of x in (3), we find

$$y=5.$$

2. Again, suppose we have given, to find x, y, and z, the three equations,

$$2x+4y-3z=22, \qquad (1)$$
$$4x-2y+5z=18, \qquad (2)$$
$$6x+7y-\ z=63. \qquad (3)$$

From equation (3) we obtain

$$z=6x+7y-63. \qquad (4)$$

Substituting this value of z, in (1) and (2), they will become

$$2x+4y-3(6x+7y-63)=22, \qquad (5)$$
$$4x-2y+5(6x+7y-63)=18. \qquad (6)$$

Equations (5) and (6) become, after expanding, transposing, and uniting terms,

$$16x+17y=167, \qquad (7)$$

$$34x+33y=333. \qquad (8)$$

Equation (7) gives

$$x=\frac{167-17y}{16}. \qquad (9)$$

This value of x, substituted in (8), gives

$$\frac{34(167-17y)}{16}+33y=333. \qquad (10)$$

Equation (10), when solved as a simple equation of one unknown quantity, gives

$$y=7.$$

Substituting this value of y in (9), we find

$$x=3.$$

Using these values of x and y in (4), we obtain

$$z=4.$$

This method of eliminating may be comprehended in the following

RULE.

Having found the value of one of the unknown quantities, from either of the given equations, in terms of the other unknown quantities; substitute it for that unknown quantity in the remaining equations, and we shall thus obtain a new system of equations one less in number than those given. Operate with these new equations as with the first, and so continue until we find one single equation with but one unknown quantity, which will then become known.

EXAMPLES.

3. Given $\left\{\begin{aligned} x-w&=50, &(1)\\ 3y-x&=120, &(2)\\ 2z-y&=120, &(3)\\ 3w-z&=195, &(4)\end{aligned}\right\}$ to find w, x, y, and z.

From (1) we find

$$w=x-50. \qquad (5)$$

This value of w, substituted in (4), gives

$$3(x-50)-z=195, \text{ or } 3x-z=345. \qquad (6)$$

Equation (6) gives

$$z=3x-345. \qquad (7)$$

This value of z, substituted in (3), gives

$$2(3x-345)-y=120, \text{ or } 6x-y=810. \qquad (8)$$

Equation (8) gives

$$y=6x-810. \qquad (9)$$

This value of y, substituted in (2), gives

$$3(6x-810)-x=120, \qquad (10)$$

or $$17x=2550. \qquad (11)$$

Therefore, $$x=150.$$

This value of x, causes (9) to become

$$y=90.$$

Using the value of x in (7), we find

$$z=105.$$

Finally, using the value of x in (5), we find

$$w=100.$$

4. Given $\left\{\begin{aligned} x+\tfrac{1}{2}y&=a, &(1)\\ y+\tfrac{1}{3}z&=a, &(2)\\ z+\tfrac{1}{4}x&=a, &(3)\end{aligned}\right\}$ to find x, y, and z.

Equation (3) gives

$$z=\frac{4a-x}{4}. \tag{4}$$

This value of z, substituted in (2), we have

$$y+\frac{4a-x}{12}=a\ . \tag{5}$$

Clearing of fractions and uniting terms, (5) becomes

$$12y-x=8a. \tag{6}$$

From (6) we find

$$x=12y-8a. \tag{7}$$

This value of x, substituted in (1), gives

$$12y-8a+\frac{y}{2}=a. \tag{8}$$

Equation (8) gives

$$25y=18a. \tag{9}$$

Therefore, $y=\frac{18a}{25}$.

This value of y, substituted in (7), gives

$$x=\frac{16a}{25}.$$

Substituting for x, in (4), its value just found, we have

$$z=\frac{21a}{25}.$$

Hence, collecting values, we have

$$\left.\begin{array}{l} x=\frac{16}{25}a, \\ y=\frac{18}{25}a, \\ z=\frac{21}{25}a. \end{array}\right\}$$

We may observe that if a is any multiple of 25, the above value of x, y, and z, will be integers.

Thus calling $a=25$, we have

$$x=16,\ y=18, \text{ and } z=21.$$

If $a=50$, we shall have

$$x=32,\ y=36, \text{ and } z=42.$$

THIRD METHOD.

ELIMINATION BY COMPARISON.

(72.) There is still another method of elimination.

We may also eliminate one of the unknown quantities of two equations, by the following process:

1. Take the two equations,

$$5y-3x=-14, \qquad (1)$$
$$3y+4x=38. \qquad (2)$$

If we, for a moment, consider y as a known quantity, we may then, from each of these equations, find the value of x by Rule under Art. 63.

We thus find

$$x=\frac{14+5y}{3}, \qquad (3)$$

$$x=\frac{38-3y}{4}. \qquad (4)$$

Putting these two values of x equal to each other, we have

$$\frac{14+5y}{3}=\frac{38-3y}{4}. \qquad (5)$$

Clearing (5) of fractions, it becomes

$$56+20y=114-9y. \qquad (6)$$

Transposing and uniting terms, we find

$$29y=58;$$

Therefore, $$y=2.$$

2. As a second example we will take three equations, with three unknowns, as follows:

$$x+2y+3z=43, \qquad (1)$$
$$3x-y+2z=24, \qquad (2)$$
$$5x+3y-4z=14. \qquad (3)$$

Considering x and y as known quantities, we readily deduce from these equations,

$$z=\frac{43-x-2y}{3}, \qquad (4)$$
$$z=\frac{24-3x+y}{2}, \qquad (5)$$
$$z=\frac{-14+5x+3y}{4}. \qquad (6)$$

Putting the first value of z equal to each of the other values, we have these two equations:

$$\frac{43-x-2y}{3}=\frac{24-3x+y}{2}, \qquad (7)$$
$$\frac{43-x-2y}{3}=\frac{-14+5x+3y}{4}. \qquad (8)$$

Equations (7) and (8), being cleared of fractions and properly transposed, give

$$7x-7y=-14, \qquad (9)$$
$$19x+17y=214. \qquad (10)$$

The rest of the labor of solution is similar to the first example, inasmuch as we have two equations and two unknowns, x and y; the z having been eliminated.

From (9) and (10) we find

$$y=\frac{14+7x}{7}=2+x, \qquad (11)$$

$$y=\frac{214-19x}{17}. \qquad (12)$$

Equating these values of y, we shall eliminate y, and thus find

$$2+x=\frac{214-19x}{17}. \qquad (13)$$

Clearing of fractions, and transposing, we have

$$36x=180;$$

Consequently $x=5$.

If we substitute this value of x in (11) we shall find

$$y=2+x=2+5=7.$$

Again, substituting the values of both x and y, in (4), we shall find

$$z=\frac{43-x-2y}{3}=\frac{43-5-14}{3}=8.$$

The foregoing method of elimination is comprised in the following

RULE.

I. Find from each of the given equations, the value of one of the unknown quantities, by Rule under Art. 63, *on the supposition that the other quantities are known.*

II. Then equate these different expressions of the value of the unknown, thus found, and we shall thus have a number of equations one less than were first given; and they will also contain a number of unknown quantities one less than at first.

III. Operating with these new equations, as was done with the given equations, we can again reduce their number one; and *continuing this process, we shall finally have but one equation containing but one unknown quantity, which will then become known.*

EXAMPLES.

3. Given $\left\{\begin{array}{rr} 7x+5y+2z=79, & (1) \\ 8x+7y+9z=122, & (2) \\ x+4y+5z=55, & (3) \end{array}\right\}$ to find x, y, and z.

By Rule under Art. 63, we find, by using (1), (2), and (3),

$$x=\frac{79-5y-2z}{7}, \qquad (4)$$

$$x=\frac{122-7y-9z}{8}, \qquad (5)$$

$$x=55-4y-5z. \qquad (6)$$

Equating (4) and (6), and (5) and (6), we have

$$\frac{79-5y-2z}{7}=55-4y-5z, \qquad (7)$$

$$\frac{122-7y-9z}{8}=55-4y-5z. \qquad (8)$$

When cleared of fractions, (7) and (8) become

$$79-5y-2z=385-28y-35z,$$

$$122-7y-9z=440-32y-40z.$$

Transposing and uniting terms, we have

$$23y+33z=306, \qquad (9)$$

$$25y+31z=318. \qquad (10)$$

Equations (9) and (10) give

$$y=\frac{306-33z}{23}, \qquad (11)$$

$$y=\frac{318-31z}{25}. \qquad (12)$$

Equating (11) and (12), we have

$$\frac{306-33z}{23}=\frac{318-31z}{25}, \qquad (13)$$

which reduced, gives

$$z=3.$$

This value of z substituted in (11), gives

$$y=9.$$

And these values of z and y, substituted in (6), give

$$x=4.$$

4. Given $\left\{\begin{array}{l} \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=62, \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=47, \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=38, \end{array}\right\}$ to find x, y, and z.

These equations, when cleared of fractions, become

$$6x+4y+3z=744, \qquad (1)$$

$$20x+15y+12z=2820, \qquad (2)$$

$$15x+12y+10z=2280. \qquad (3)$$

From (1), (2), and (3), we find

$$z=\frac{744-6x-4y}{3}, \qquad (4)$$

$$z=\frac{2820-20x-15y}{12}, \qquad (5)$$

$$z=\frac{2280-15x-12y}{10}. \qquad (6)$$

Equating (4) with (5), and (4) with (6), we have

$$\frac{744-6x-4y}{3}=\frac{2820-20x-15y}{12}, \qquad (7)$$

$$\frac{744-6x-4y}{3}=\frac{2280-15x-12y}{10}. \qquad (8)$$

Equations (7) and (8), when reduced, become

$$4x+y=156, \qquad (9)$$

$$15x+4y=600. \qquad (10)$$

Equations (9) and (10) give

$$y=156-4x, \qquad (11)$$

$$y=\frac{600-15x}{4}. \qquad (12)$$

Equating (11) and (12), we have

$$156-4x=\frac{600-15x}{4}. \qquad (13)$$

This reduced, gives

$$x=24.$$

Having found x, we readily find y and z to be

$$y=60;\ z=120.$$

All equations of the first degree, containing any number of unknown quantities, can be solved by either of the Rules under Articles 69, 71, and 72, or by a combination of the same.

The student must exercise his own judgment, as to the choice of the above Rules. In very many cases, he will discover many short processes, which depend upon the particular equations given.

(73.) We will now solve a few equations, and shall endeavor to effect their solution in the simplest manner possible.

1. Given $\left\{\begin{array}{l} 6x+5y=128, \\ 3x+4y=\ 88, \end{array}\right\}$ to find the values of x and y.

Adding the two equations, and dividing the sum by 9, we find

$$x+y=24. \qquad (1)$$

Multiplying (1) by 3, and subtracting the result from the second of the given equations, we have

$$y=16. \qquad (2)$$

Subtraction (2) from (1), we get

$$x=8.$$

2. Given $\left\{\begin{array}{ll} x+y=a, & (1) \\ y+z=b, & (2) \\ z+x=c, & (3) \end{array}\right\}$ to find x, y, and z.

Dividing the sum of these three equations by 2, we find

$$x+y+z=\frac{a+b+c}{2}. \qquad (4)$$

From (4) subtracting, successively, (2), (3), and (1), we find

$$\left.\begin{aligned} x&=\frac{a+c-b}{2},\\ y&=\frac{a-c+b}{2},\\ z&=\frac{-a+b+c}{2}. \end{aligned}\right\} \qquad (A)$$

Equations (1), (2), and (3), of this last example are so related that if in (1) we change x to y, y to z, and a to b, it will correspond with (2). Again, if in (2) we change y to z, z to x, and b to c, it will correspond with (3). Also, if in (3) we change z to x, x to y, and c to a, it will give (1), from which we first started. In each change we have advanced the letters one place lower in the alphabetical scale, observing that when we wish to change the last letters of the series, as z or c, we must change them respectively to x and a, the first of the series.

Since the above changes can be made with the primitive equations (1), (2), (3), without altering the conditions of the question, it follows that the same changes can be made in any of the equations derived from those. Thus, executing those changes in equations (A), we find that the first is changed into the second, the second into the third, and the third in turn is changed into the first.

3. Given $\left\{\begin{aligned} \frac{1}{x}+\frac{1}{y}&=a, \quad (1)\\ \frac{1}{y}+\frac{1}{z}&=b, \quad (2)\\ \frac{1}{z}+\frac{1}{x}&=c, \quad (3) \end{aligned}\right\}$ to find x, y, and z.

If we take the sum of these three equations, we shall obtain

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=a+b+c. \qquad (4)$$

Now, subtracting twice (2) from (4), we have

$$\frac{2}{x}=a+c-b. \qquad (5)$$

In a similar manner subtracting twice (3) and (1), successively, from (4), we find

$$\frac{2}{y}=a-c+b, \qquad (6)$$

$$\frac{2}{z}=-a+b+c. \qquad (7)$$

Equations (5), (6), and (7), readily give

$$\left.\begin{aligned} x&=\frac{2}{a+c-b},\\ y&=\frac{2}{a-c+b},\\ z&=\frac{2}{-a+b+c}.\end{aligned}\right\} \qquad (B)$$

The letters in this example will admit of the same changes as those pointed out in the last example. Indeed, the only difference between the two examples is, that the unknown quantities in the one example are the reciprocals of those in the other. Consequently the expressions for x, y, and z, as given by equations (B), ought to be the reciprocals of those given by equations (A), which we find to be really the case.

4. Given $\left\{\begin{array}{ll} u+x+y=13, & (1)\\ u+x+z=17, & (2)\\ u+y+z=18, & (3)\\ x+y+z=21, & (4)\end{array}\right\}$ to find u, x, y, and z.

Dividing the sum of these four equations by 3, we obtain

$$u+x+y+z=23. \qquad (5)$$

From (5), subtracting successively, (4), (3), (2), and (1), we find

$$\left.\begin{aligned} u&=2,\\ x&=5,\\ y&=6,\\ z&=10. \end{aligned}\right\}$$

5. A man left a sum of money which was divided among 4 children, so that the share of the first was ½ the sum of the shares of the other three; the share of the second ⅓ of the sum of the other three; and the share of the third ¼ of the sum of the other three; and it was also found that the share of the first exceeded that of the last by $7 (*a*). What was the share of each child?

Let u, x, y, and z represent respectively their shares.

Then by the question we have these four conditions:

$$u=\frac{x+y+z}{2}, \qquad (1)$$

$$x=\frac{u+y+z}{3}, \qquad (2)$$

$$y=\frac{u+x+z}{4}. \qquad (3)$$

$$u-a=z. \qquad (4)$$

These equations, cleared of fractions, become

$$\begin{aligned} 2u&=x+y+z, &(5)\\ 3x&=u+y+z, &(6)\\ 4y&=u+x+z, &(7)\\ u-a&=z. &(8) \end{aligned}$$

If we add u to (5), x to (6), y to (7), their right-hand mem bers will each become $u+x+y+z$; consequently, their left-hand members will be equal, and we shall have

$$\left.\begin{array}{l}3u=4x,\\ 3u=5y.\end{array}\right\} \qquad (9)$$

Therefore,

$$x=\frac{3u}{4};\ y=\frac{3u}{5}.$$

Which value causes (5) to become

$$2u=\frac{3u}{4}+\frac{3u}{5}+z. \qquad (10)$$

This gives

$$z=\frac{13u}{20}. \qquad (11)$$

Substituting this value of z in (8), we find

$$u=\frac{20a}{7}.$$

This value of u readily makes known the value of the other letters. Collecting the values we have

$$u=\frac{20a}{7}. \qquad x=\frac{15a}{7}.$$

$$y=\frac{12a}{7}. \qquad z=\frac{13a}{7}.$$

Now, it is obvious that the least value which can be given to a so as to avoid fractions is 7. It is also obvious that a being any multiple of 7, the values of u, x, y, and z, will be integral.

When $a=7$, we find

$$u=20,$$
$$x=15,$$
$$y=12,$$
$$z=13.$$

And, in all cases, the values are in the ratio of these last values.

From what has been said and done, the student will be able to solve the following problems without any serious difficulty.

6. Given $\left\{\begin{array}{r} 3x+2y=118, \\ x+5y=191, \end{array}\right\}$ to find x and y. *Ans.* $\left\{\begin{array}{l} x=16. \\ y=35. \end{array}\right.$

7. A. and B. possess together a fortune of \$570. If A.'s fortune were 3 times, and B.'s five times as great as each really is, then they would have together \$2350. How much had each? *Ans.* A. \$250, B. \$320.

8. Find two numbers of the following properties: When the first is multiplied by 2, the other by 5, and both products added together, the sum is $=31$; on the other hand, if the first be multiplied by 7, and the second by 4, and both products added together, we shall obtain 68.

Ans. The first is 8, and the second is 3.

9. A. owes \$1200, B. \$2550; but neither has enough to pay his debts. Lend me, said A. to B., $\frac{1}{8}$ of your fortune, and I shall be enabled to pay my debts. B. answered, I can discharge my debts if you will lend me $\frac{1}{6}$ of yours. What was the fortune of each? *Ans.* A.'s fortune is \$900, and that of B.'s \$2400.

10. There is a fraction, such, that if 1 be added to the numerator, its value $=\frac{1}{3}$, and if 1 be added to the denominator, its value $=\frac{1}{4}$. What fraction is it? *Ans.* $\frac{4}{15}$.

11. A., B., C., owe together \$2190, and neither of them can alone pay this sum; but when they unite, it can be done in the following ways: first, by B.'s putting $\frac{3}{7}$ of his property to all of A.'s; secondly, by C.'s putting $\frac{5}{9}$ of his property to all of B.'s; or thirdly, by A.'s adding $\frac{2}{3}$ of his property to C.'s. How much did each possess? *Ans.* A. \$1530, B. \$1540, and C \$1170

12. Three masons, A., B., C., are to build a wall. A. and B., jointly, could build this wall in 12 days; B. and C. could accomplish it in 20 days; A. and C. could do it in 15 days. What time would each take to do it alone in? And in what time will they finish it if all three work together?

Ans. A. requires 20 days, B. 30, and C. 60; all three together require 10 days.

13. If A. and B., with C. working half time, can build a wall in 21 days; B. and C., with D. working half time, in 24 days; C. and D., with A. working half time, in 28 days; D. and A., with B. working half time, in 32 days; in what time would it be built by all together, and by each alone?

Ans.
All would require 16 days.
A. " " $52\frac{1}{2}$ "
B. " " $57\frac{27}{29}$ "
C. " " $44\frac{4}{19}$ "
D. " " 280 "

14. A cistern containing 210 buckets, may be filled by 2 pipes. By an experiment, in which the first was open 4, and the second 5 hours, 90 buckets of water were obtained. By another experiment, when the first was open 7, and the other $3\frac{1}{2}$ hours, 126 buckets were obtained. How many buckets does each pipe discharge in an hour? And in what time will the cistern be full, when the water flows from both pipes at once?

Ans. The first pipe discharges 15, and the second, 6 buckets; it will require 10 hours for them to fill the cistern.

15. According to Vitruvius, Hiero's crown weighed 20 lbs., and lost $1\frac{1}{4}$ lbs., nearly, in water. Let it be assumed that it consisted of gold and silver only, and that 20 lbs. of gold lose 1 lb. in water, and 10 lbs. of silver, in like manner, lose 1 lb. How much gold, and how much silver did this crown contain?

Ans. 15 lbs. of gold, and 5 lbs. of silver.

16. A person has two large pieces of iron whose weights are required. It is known that $\frac{2}{5}$ of the first piece weighs 96 lbs. less than $\frac{3}{4}$ of the other piece; and that $\frac{5}{8}$ of the other piece weighs exactly as much as $\frac{4}{9}$ of the first. How much did each of these pieces weigh?

Ans. The first weighs 720, the second 512 lbs.

17. Two persons, A. and B., were eating apples. Says A. to B., if you give me 2 of yours, I shall have as many as you. If you give me 2 of yours, answered B., I shall have twice as many as you. How many had each? *Ans.* A. had 10, and B. 14.

18. It is required to find three numbers, such, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third, shall together make 46; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third, shall together make 35; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third, shall together make $28\frac{1}{3}$. *Ans.* 12, 60, and 80.

19. A person expends 30 cents in apples and pears, giving a cent for four apples, and a cent for five pears. He afterwards parts with half his apples, and one-third of his pears, the cost of which was thirteen cents. How many did he buy of each?

Ans. 72 apples, and 60 pears.

20. The ages of three men, A., B., and C., are such, that $\frac{1}{2}$ of A.'s, $\frac{1}{4}$ of B.'s, and $\frac{1}{3}$ of C.'s make 80 years; $\frac{1}{3}$ of A.'s, $\frac{1}{2}$ of B.'s, and $\frac{1}{5}$ of C.'s make 78 years; and $\frac{1}{6}$ of A.'s, $\frac{1}{8}$ of B.'s, and $\frac{1}{6}$ of C.'s make 35 years. Required the age of each.

Ans. A.'s age was 60 years.
B.'s age was 80 years.
C.'s age was 90 years.

21. A wine merchant has two kinds of wine: the first kind is worth \$2 per bottle, the second is \$$1\frac{1}{2}$. He wishes to make a mixture of 120 bottles, which shall be worth \$$1\frac{2}{3}$ per bottle. How many bottles must he take of each kind?

Ans. 40 bottles of first kind.
80 bottles of second kind.

22. The fortunes of four persons, A., B., C., and D., are determined by the following conditions: A. and B. have together \$1000; C. and D. \$1200; A.'s fortune is double that of C.'s, and D.'s three times that of B.'s. How much had each?

Ans.
A. had \$720.
B. had \$280.
C. had \$360.
D. had \$840.

23. A man, who undertook to transport some porcelain vases of three different sizes, contracted that he would pay as much for each vessel that he broke, as he received for those which he delivered safe.

He had committed to him 2 small vases, 4 of a middle size, and 9 large ones: he broke the middle-sized ones, delivered all the others safe, and received 28 cents.

There were afterwards committed to him 7 small vases, 3 of the middle size, and 5 of the large ones: this time he broke the large ones, delivered the others, and received only 3 cents.

Lastly, he took charge of 9 small vases, 10 middle-sized ones, and 11 large ones; this time he also broke the large ones, and delivered the others, and received only 4 cents.

How much was paid for carrying a vase of each kind?

Ans.
He received 2 cents for carrying a small one.
" 3 " " a middle-sized one.
" 4 " " a large one.

24. Find four numbers, so that the first together with half the second, may be 357; the second with one-third of the third, may be 476; the third with one-fourth of the fourth, may be 595; and the fourth with one-fifth of the first, may be 714.

Ans.
The first $=190$.
The second $=334$.
The third $=426$.
The fourth $=676$.

(74.) PROBLEMS GENERALIZED.

Problem 10 is susceptible of a beautiful generalization, as follows:

25. "There is a fraction, such, that if a be added to the numerator, its value becomes $\frac{1}{m}$, but if a were added to the denominator its value would be $\frac{1}{n}$. What is this fraction?"

Let x denote its numerator, and y its denominator.

Then by the conditions we shall have

$$\frac{x+a}{y}=\frac{1}{m}, \qquad (1)$$

$$\frac{x}{y+a}=\frac{1}{n}. \qquad (2)$$

These equations readily give

$$x=\frac{a(m+1)}{n-m}.$$

$$y=\frac{m(n+a)}{n-m}.$$

Hence, the fraction is found, by dividing x by y, to be

$$\frac{x}{y}=Fraction=\frac{a(m+1)}{m(n+a)}.$$

That this result may agree with Prob. 10, we must take $a=1$, $m=3$, and $n=4$. Substituting these values in the above general result, we find $Fraction=\frac{4}{15}$.

Problem 12 may be generalized as follows:

26. "Three agents, A., B., C., are employed on a certain work. A. and B. jointly could do this work in a days; B. and C. could do it in b days; C. and A. could do it in c days. What time

would each alone require to do it? And in what time will they finish it if all three work together?"

If x, y, z denote the number of days it will require for A., B., C., respectively, to perform the work, then will $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ denote the fractional parts performed by each respectively in one day. Hence, the fractional part performed by A. and B. jointly is $\frac{1}{x}+\frac{1}{y}$. But since they together perform the whole work in a days, they will perform in one day the fractional part denoted by $\frac{1}{a}$. Reasoning in a similar way in reference to B. and C. jointly, and C. and A. jointly, we shall have the following conditions:

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{a}=\frac{bc}{abc}, \qquad (1)$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{b}=\frac{ac}{abc}, \qquad (2)$$

$$\frac{1}{z}+\frac{1}{x}=\frac{1}{c}=\frac{ab}{abc}. \qquad (3)$$

Half the sum of these three equations is

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{ab+bc+ca}{2abc}. \qquad (4)$$

This expresses the fractional part of the work performed by all in one day; consequently, the time required for all to do the work is the reciprocal of this expression, that is, they will all require a number of days denoted by

$$\frac{2abc}{ab+bc+ca}=\text{time for all.}$$

Subtracting successively (2), (3), and (1) from (4), we find

$$\frac{1}{x}=\frac{ab+bc-ca}{2abc}, \tag{5}$$

$$\frac{1}{y}=\frac{bc+ca-ab}{2abc}, \tag{6}$$

$$\frac{1}{z}=\frac{ca+ab-bc}{2abc}. \tag{7}$$

The reciprocals of these give

$$x=\frac{2abc}{ab+bc-ca}=\text{A.'s time.}$$

$$y=\frac{2abc}{bc+ca-ab}=\text{B.'s time.}$$

$$z=\frac{2abc}{ca+ab-bc}=\text{C.'s time.}$$

That these general results may agree with Problem 12, we must take $a=12$, $b=20$, and $c=15$. Using these values, we find

$$\frac{2abc}{ab+bc+ca}=10, \text{ number of days for all.}$$

$$\frac{2abc}{ab+bc-ca}=20, \quad \text{“} \quad \text{“} \quad \text{“ A.}$$

$$\frac{2abc}{bc+ca-ab}=30, \quad \text{“} \quad \text{“} \quad \text{“ B.}$$

$$\frac{2abc}{ca+ab-bc}=60, \quad \text{“} \quad \text{“} \quad \text{“ C.}$$

Before dismissing this Problem, we remark that equations (1), (2), (3), of Problem 3, will agree with this, if we change a, b, c into their reciprocals. Then it follows that changing a, b, c into their reciprocals in the values of x, y, and z, of Prob. 3, they must agree with the values of x, y, and z of this Problem. Let us actually make these changes:

$$\frac{2}{c+a-b} \text{ becomes } \qquad \frac{2}{\frac{1}{c}+\frac{1}{a}-\frac{1}{b}}=\frac{2abc}{ab+bc-ca}.$$

$$\frac{2}{a+b-c} \text{ becomes } \qquad \frac{2}{\frac{1}{a}+\frac{1}{b}-\frac{1}{c}}=\frac{2abc}{bc+ca-ab}.$$

$$\frac{2}{b+c-a} \text{ becomes } \qquad \frac{2}{\frac{1}{b}+\frac{1}{c}-\frac{1}{a}}=\frac{2abc}{ca+ab-bc}.$$

We will also generalize Problem 13, as follows:

27. "If A. and B., with C. working half time, can do a piece of work in a days; B. and C., with D. working half time, in b days; C. and D., with A. working half time, in c days; D. and A., with B. working half time, in d days; how many would all require, and how many for each separately?"

If we denote by w, x, y, and z the number of days respectively for each alone to perform the work, we may, by a process of reasoning similar to that employed in the last Problem, obtain the following conditions:

$$\frac{1}{w}+\frac{1}{x}+\frac{1}{2y}=\frac{1}{a}, \tag{1}$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{2z}=\frac{1}{b}, \tag{2}$$

$$\frac{1}{y}+\frac{1}{z}+\frac{1}{2w}=\frac{1}{c}, \tag{3}$$

$$\frac{1}{z}+\frac{1}{w}+\frac{1}{2x}=\frac{1}{d}. \tag{4}$$

Dividing the sum of these four equations by $\frac{5}{2}$, we find

$$\frac{1}{w}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\tfrac{2}{5} \text{ of } \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right), \tag{5}$$

which expresses the fractional part performed by all in one day. Consequently, they would together require a number of days denoted by

$$\frac{1}{\tfrac{2}{5} \text{ of } \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)}=\frac{5abcd}{2bcd+2cda+2dab+2abc}.$$

Subtracting (2) from (1), we have

$$\frac{1}{w}-\frac{1}{2y}-\frac{1}{2z}=\frac{1}{a}-\frac{1}{b}. \qquad (6)$$

To (6) adding the half of (3), we find

$$\frac{5}{4w}=\frac{1}{a}-\frac{1}{b}+\frac{1}{2c}. \qquad (7)$$

Dividing by $\frac{5}{4}$, we find

$$\frac{1}{w}=\frac{4}{5}\text{ of }\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{2c}\right),$$

which denotes the fractional part performed by A. in one day. Consequently A. would require a number of days denoted by

$$\frac{1}{\frac{4}{5}\text{ of }\left(\frac{1}{a}-\frac{1}{b}+\frac{1}{2c}\right)}=\frac{5abc}{2ab+4bc-4ca}.$$

The other expressions for B., C., and D. may be deduced from this by simply permuting the letters, as was explained under Problems 2 and 3. Making such permutations, we shall obtain

$$\frac{5abc}{2ab+4bc-4ca}=\text{A.'s time.}$$

$$\frac{5bcd}{2bc+4cd-4db}=\text{B.'s time.}$$

$$\frac{5cda}{2cd+4da-4ac}=\text{C.'s time.}$$

$$\frac{5dab}{2da+4ab-4bd}=\text{D.'s time.}$$

For still further illustrations of this important subject of *Generalization* of Problems, see Chapter VII.

CHAPTER IV.

INVOLUTION, EVOLUTION, IRRATIONAL AND IMAGINARY QUANTITIES.

INVOLUTION.

(75.) The process of raising a quantity to any proposed power, is called INVOLUTION.

When the quantity is a single letter, it may be involved by placing the number denoting the power above it a little to the right. (Art. 11.)

After the same manner we may represent the power of any quantity, by inclosing it within a parenthesis, and then treating it as a single letter. Thus,

The second power of $mx=(mx)^2$,
The third power of $a+b=(a+b)^3$,
The fourth power of $3m+y=(3m+y)^4$,
&c. &c.

CASE I.

(76.) To involve a monomial, we obviously have this

RULE.

I. Raise the coefficient to the required power, by actual multiplication.

II. Raise the different letters to the required power by multiplying the exponents, which they already have, by the number denoting the power; observing that if no exponent is written, then 1 is always understood. To this power prefix the power of the coefficient.

NOTE.—If the quantity to be involved is negative, the signs of the *even* powers must be positive, and the signs of the *odd* powers negative. (Art. 23.)

EXAMPLES.

1. What is the square of $3ax^3$?

Here the square of 3 equals

$$3^2=3\times 3=9.$$

Considering the exponent of a, in the expression ax^3, as 1, we find a^2x^6 for the square of ax^3.

Therefore we have

$$(3ax^3)^2=9a^2x^6.$$

2. What is the fifth power of $-2ab^3$?

Ans. $(-2ab^3)^5=-32a^5b^{15}$.

3. What is the fourth power of $-\frac{1}{3}xy^{-2}$?

Ans. $(-\frac{1}{3}xy^{-2})^4=\frac{1}{81}x^4y^{-8}$,

which by Art. 32, is the same as $\dfrac{x^4}{81y^8}$.

4. What is the seventh power of $-a^{-\frac{1}{2}}x$?

Ans. $-a^{-\frac{7}{2}}x^7=-\dfrac{x^7}{a^{\frac{7}{2}}}$.

5. What is the third power of x^3y^{-1}? *Ans.* $x^9y^{-3}=\dfrac{x^9}{y^3}$.

6. What is the nth power of $-2x^{-3}y^2$?

Ans. $\pm 2^nx^{-3n}y^{2n}=\pm\dfrac{2^ny^{2n}}{x^{3n}}$.

7. What is the square of $-7x^{-1}y^{-3}$? *Ans.* $49x^{-2}y^{-6}=\dfrac{49}{x^2y^6}$.

8. What is the third power of $-\frac{1}{5}x^3y^{-5}$?

Ans. $-\frac{1}{125}x^9y^{-15}=-\frac{x^9}{125y^{15}}$.

9. What is the seventh power of $-m^{\frac{1}{3}}xz^{-1}$?

Ans. $-m^{\frac{7}{3}}x^7z^{-7}$.

10. What is the fourth power of $-\frac{2}{3}n^{-2}y^3$? Ans. $\frac{16}{81}n^{-8}y^{12}$.

11. What is the square of $-3a^2$? Ans. $9a^4$.

12. What is the cube of $-4a^{\frac{1}{2}}x^2$?

Ans. $-64a^{\frac{3}{2}}x^6$.

13. What is the fifth power of $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$? Ans. $a^{\frac{5}{2}}b^{\frac{5}{3}}c^{\frac{5}{4}}$.

14. What is the cube of $-a^{-1}$? Ans. $-a^{-3}=-\frac{1}{a^3}$.

15. What is the fourth power of $-a^{-\frac{1}{4}}$?

Ans. $a^{-1}=\frac{1}{a}$.

16. What is the square of $\frac{3a^{-2}}{4b}$? Ans. $\frac{9a^{-4}}{16b^2}=\frac{9}{16a^4b^2}$.

17. What is the eleventh power of $a^{-1}bc^{-2}$?

Ans. $a^{-11}b^{11}c^{-22}=\frac{b^{11}}{a^{11}c^{22}}$.

18. What is the cube of $\frac{3x}{4y^{-1}}$? Ans. $\frac{27x^3}{64y^{-3}}=\frac{27x^3y^3}{64}$.

19. What is the cube of $\pm m^2x^{\frac{1}{2}}$? Ans. $\pm m^6x^{\frac{3}{2}}$.

20. What is the square of $\pm a^{\frac{1}{3}}b^{-\frac{1}{4}}$?

Ans. $a^{\frac{2}{3}}b^{-\frac{1}{2}}=\frac{a^{\frac{2}{3}}}{b^{\frac{1}{2}}}$.

21. What is the square of $-\frac{4}{5}\,a^{-\frac{1}{2}}$?

Ans. $\frac{16}{25a}$.

22. What is the fifth power of $-2a^2b^3$? Ans. $-32a^{10}b^{15}$.

CASE II.

(77.) To involve a Polynomial.

At present we will content ourselves by involving compound expressions, by actual multiplication, according to Rule under Art. 27.

EXAMPLES.

1. Find the second power of $x+y-z$.

$$\begin{array}{rl}
 & x+y-z \\
 & x+y-z \\
\cline{2-2}
 & x^2+xy-xz \\
 & \quad +xy \qquad +y^2-yz \\
 & \qquad -xz \qquad -yx+z^2 \\
\cline{2-2}
\textit{Ans.} & =x^2+2xy-2xz+y^2-2yx+z^2.
\end{array}$$

2. Find the fifth power of $a+b$, as well as the lower powers of the same.

$$\begin{array}{rl}
(a+b)^1= & a+b. \\
 & a+b \\
\cline{2-2}
 & a^2+ab \\
 & \quad +ab+b^2 \\
\cline{2-2}
(a+b)^2= & a^2+2ab+b^2. \\
 & a+b \\
\cline{2-2}
 & a^3+2a^2b+ab^2 \\
 & \qquad a^2b+2ab^2+b^3 \\
\cline{2-2}
(a+b)^3= & a^3+3a^2b+3ab^2+b^3. \\
 & a+b \\
\cline{2-2}
 & a^4+3a^3b+3a^2b^2+ab^3 \\
 & \qquad a^3b+3a^2b^2+3ab^3+b^4 \\
\cline{2-2}
(a+b)^4= & a^4+4a^3b+6a^2b^2+4ab^3+b^4. \\
 & a+b \\
\cline{2-2}
 & a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
 & \qquad a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\
\cline{2-2}
(a+b)^5= & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.
\end{array}$$

3. Find the fifth power of $a-b$.

$$\begin{array}{rl}
(a-b)^1= & a-b. \\
 & a-b \\ \cline{2-2}
 & a^2-ab \\
 & \quad -ab+b^2 \\ \cline{2-2}
(a-b)^2= & a^2-2ab+b^2. \\
 & a\ -b \\ \cline{2-2}
 & a^3-2a^2b+ab^2 \\
 & \quad -a^2b+2ab^2-b^3 \\ \cline{2-2}
(a-b)^3= & a^3-3a^2b+3ab^2-b^3. \\
 & a\ -b \\ \cline{2-2}
 & a^4-3a^3b+3a^2b^2-\ ab^3 \\
 & \quad -\ a^3b+3a^2b^2-3ab^3+b^4 \\ \cline{2-2}
(a-b)^4= & a^4-4a^3b+6a^2b^2-4ab^3+b^4. \\
 & a\ -b \\ \cline{2-2}
 & a^5-4a^4b+\ 6a^3b^2-\ 4a^2b^3+ab^4 \\
 & \quad -\ a^4b+\ 4a^3b^2-\ 6a^2b^3+4ab^4-b^5 \\ \cline{2-2}
(a-b)^5= & a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5.
\end{array}$$

4. What is the cube of $a-x$? *Ans.* $a^3-3a^2x+3ax^2-x^3$

5. What is the square of $m+n-x$?
Ans. $m^2+2mn-2mx+n^2-2nx+x^2$

6. What is the fourth power of $3x-2y$?
Ans. $81x^4-216x^3y+216x^2y^2-96xy^3+16y^4$.

7. What is the square of $a+b$? *Ans.* $a^2+2ab+b^2$.

8. What is the square of $a+b+c$?
Ans. $a^2+2ab+2ac+b^2+2bc+c^2$.

9. What the square of x^2+x? *Ans.* $x^4+2x^3+x^2$.

10. What is the square of x^2+x+1?
Ans. $x^4+2x^3+3x^2+2x+1$.

11. What is the square of $a^{\frac{1}{2}}+b^{\frac{1}{2}}$? *Ans.* $a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b$.

12. What is the square of $\frac{1}{2}a-\frac{1}{2}b$? *Ans.* $\frac{1}{4}a^2-\frac{1}{2}ab+\frac{1}{4}b^2$.

13. What is the cube of $x-y$? *Ans.* $x^3-3x^2y+3xy^2-y^3$.

14. What is the 4th power of $a-a^{-1}$?
Ans. $a^4-4a^2+6-4a^{-2}+a^{-4}$.

15. What is the square of $a^{-\frac{1}{2}}-a^{\frac{1}{2}}$? *Ans.* $a^{-1}-2+a$.

16. What is the square of $a^3 \pm a^{-3}$? *Ans.* $a^6 \pm 2+a^{-6}$.

17. What is the cube of $a^{\frac{1}{3}}-a^{-\frac{1}{3}}$?
Ans. $a-3a^{\frac{1}{3}}+3a^{-\frac{1}{3}}-a^{-1}$.

EVOLUTION.

(78.) Evolution is the extracting of roots, or the reverse process of Involution.

When the quantity whose root is to be found is a single letter, the operation is denoted by giving it a fractional exponent, the denominator of which denotes the degree of the root.

And in the same manner we may denote the extraction of a root of any quantity or expression, by inclosing it in a parenthesis, and then treating it as a single letter. Thus,

The second root of $my=(my)^{\frac{1}{2}}$,

The third root of $x+y=(x+y)^{\frac{1}{3}}$,

The fourth root of $2x-3y=(2x-3y)^{\frac{1}{4}}$,

The nth root of $a-b=(a-b)^{\frac{1}{n}}$,

&c. &c.

CASE I.

(79.) To extract a root of a monomial, we obviously have the following

RULE.

I. Extract the required root of the coefficient by the usual arithmetical rule. (See Arithmetic.) *When the root cannot be accurately obtained, it may be denoted by means of a fractional exponent, the same as in the case of a letter.*

II. Extract the required root of the different letters, by multiplying the exponents which they already have by the fractional exponent denoting the required root. To this root prefix the root of the coefficient.

NOTE.—Since the *even* powers of all quantities, whether positive or negative, are positive, it follows that an *even* root of a negative quantity is *impossible*, and an *even* root of a positive quantity is either positive or negative. We also infer that an *odd* root of any quantity has the same sign as the quantity itself.

EXAMPLES.

1. What is the square root of $64a^2b^4x^6$?

In this example, the square root of the coefficient, 64, is ± 8, which is read, $+8$, or -8.

And,

$$(a^2b^4x^6)^{\frac{1}{2}}=ab^2x^3.$$

Therefore, $(64a^2b^4x^6)^{\frac{1}{2}}=\pm 8ab^2x^3$.

2. What is the cube root of $64a^3x^6$? *Ans.* $4ax^2$.

3. What is the fifth root of $-32xy^2$? *Ans.* $-2x^{\frac{1}{5}}y^{\frac{2}{5}}$.

4. What is the seventh root of $-ax^{-2}$?

Ans. $-a^{\frac{1}{7}}x^{-\frac{2}{7}}=-\dfrac{a^{\frac{1}{7}}}{x^{\frac{2}{7}}}$.

5. What is the square root of $-4a^4b^2$? *Ans.* Impossible.

6. What is the cube root of $27a^3b^{12}$? *Ans.* $3ab^4$.

7. What is the fourth root of $16a^{-3}bx^{-1}$?

Ans. $\pm 2a^{-\frac{3}{4}}b^{\frac{1}{4}}x^{-\frac{1}{4}} = \pm\frac{2b^{\frac{1}{4}}}{a^{\frac{3}{4}}x^{\frac{1}{4}}}$.

8. What is the cube root of $-8a^{-6}b^{12}$? *Ans.* $-2a^{-2}b^{4}$.

9. What is the cube root of $27ab^{2}c^{3}$? *Ans.* $3a^{\frac{1}{3}}b^{\frac{2}{3}}c$.

10. What is the fourth root of $64a^{4}b^{-1}$?

Ans. $\pm 4ab^{-\frac{1}{4}} = \pm\frac{4a}{b^{\frac{1}{4}}}$.

11. What is the square root of $-4a^{2}b^{4}$? *Ans.* Impossible.

12. What is the square root of $9x^{2}y$? *Ans.* $\pm 3xy^{\frac{1}{2}}$.

13. What is the 5th root of $a^{10}bxy^{-\cdot}$? *Ans.* $a^{2}b^{\frac{1}{5}}x^{\frac{1}{5}}y^{-\frac{1}{5}}$.

14. What is the 6th root of $64a^{6}bx^{2}$? *Ans.* $\pm 2ab^{\frac{1}{6}}x^{\frac{1}{3}}$.

15. What is the 6th root of $-a^{6}$? *Ans.* Impossible.

(80.) By comparing the operations of this rule with those of the rule under Art. 76, we see that involution and evolution of monomials may both be performed by this general

RULE.

Multiply the exponents of the respective letters by the exponent denoting the power or root.

EXAMPLES.

1. What is the cube root of the second power of $8a^{3}b^{9}$?

If we first raise $8a^{3}b^{9}$ to the second power, it will become

$$(8a^{3}b^{9})^{2} = 64a^{6}b^{18}.$$

Extracting the third root, we find

$$(64a^{6}b^{18})^{\frac{1}{3}} = 4a^{2}b^{6},$$

for the result required.

Again, first extracting the cube root of $8a^3b^9$, it becomes

$$(8a^3b^9)^{\frac{1}{3}}=2ab^3.$$

Raising this to the second power, it becomes

$$(2ab^3)^2=4a^2b^6,$$

the same as before.

(81.) Hence the cube root of the square of a quantity is the same as the square of the cube root of the same quantity.

And, in general, *the nth root of the mth power of a quantity is the same as the mth power of the nth root of the same quantity.*

Therefore, $a^{\frac{4}{5}}$ may be read, the fourth power of the fifth root of a, or the fifth root of the fourth power of a.

And in the same way, $(a+b)^{\frac{3}{2}}$ is read, the third power of the square root of the sum of a and b, or the square root of the third power of the sum of a and b.

Surd quantities may be made to assume several equivalent forms, which require to be read differently. As an example, the surd $a^{-\frac{3}{5}}$ may be written six different ways, as follows:

$$1.\ \left((a^3)^{\frac{1}{5}}\right)^{-1}; \qquad 2.\ \left((a^{\frac{1}{5}})^3\right)^{-1}; \qquad 3.\ \left((a^{-1})^{\frac{1}{5}}\right)^3;$$

$$4.\ \left((a^{\frac{1}{5}})^{-1}\right)^3; \qquad 5.\ \left((a^{-1})^3\right)^{\frac{1}{5}}; \qquad 6.\ \left((a^3)^{-1}\right)^{\frac{1}{5}}.$$

These six expressions are read as follows:

1. The reciprocal of the fifth root of the third power of a.
2. The reciprocal of the third power of the fifth root of a.
3. The third power of the fifth root of the reciprocal of a.
4. The third power of the reciprocal of the fifth root of a.
5. The fifth root of the third power of the reciprocal of a.
6. The fifth root of the reciprocal of the third power of a.

Since an algebraic expression is simply an indication of certain operations to be performed, it follows that they should be read in a reverse order of the successive operations if actually performed; that is, the first operation to be performed must be the last read, and the second operation required to be performed must be read next to the last, and so on in a reverse order. Thus, in our 6th expression, we are first to cube a, then reciprocate it, and then to take the fifth root. Now it is read in the reverse order.

The student will find great assistance in reading complex algebraic expressions by paying attention to this principle. As a still further elucidation of this subject, we will give the reading of the following examples:

1. $\sqrt{\frac{1}{2}(a+b)}$. 2. $\frac{1}{2}(\sqrt{a}+\sqrt{b})$.

3. $\frac{1}{5}\times\left\{\frac{(\sqrt{3a})^{\frac{1}{3}}}{8}\right\}^4$. 4. $\frac{(\sqrt[3]{3a^{-1}})^4}{2}$.

1. The square root, of one-half, of the sum, of a and b.

2. One-half, of the sum, of the square roots, of a and b.

3. One-fifth, of the fourth power, of one-eighth, of the cube root, of the square root, of three times a.

4. One-half, of the fourth power, of the cube root, of three times the reciprocal of a.

2. What is the value of $(-3ab^2x^3)^{\frac{2}{3}}$? *Ans.* $3^{\frac{2}{3}}a^{\frac{2}{3}}b^{\frac{4}{3}}x^2$.

3. What is the value of $(4a^{-2}b^4x)^{\frac{5}{2}}$? *Ans.* $\pm 32a^{-5}b^{10}x^{\frac{5}{2}}$.

4. What is the value of $(9a^2b^4)^{\frac{3}{2}}$? *Ans.* $\pm 27a^3b^6$.

5. What is the value of $(8a^6x)^{\frac{2}{3}}$? *Ans.* $4a^4x^{\frac{2}{3}}$.

6. What is the value of $(64a^6b^6x^{12})^{\frac{3}{2}}$? *Ans.* $\pm 512a^9b^9x^{18}$.

7. What is the value of $(64a^6b^6x^{12})^{\frac{2}{3}}$? *Ans.* $16a^4b^4x^8$.

8. What is the value of $(8ab)^{-\frac{2}{3}}$? *Ans.* $4^{-1}a^{-\frac{2}{3}}b^{-\frac{2}{3}}=\frac{1}{4a^{\frac{2}{3}}b^{\frac{2}{3}}}$.

9. What is the value of $(16a^8b^{10})^{\frac{3}{2}}$? *Ans.* $\pm 64a^{12}b^{15}$.

10. What is the value of $(8a^3b^6)^{\frac{2}{3}}$? *Ans.* $4a^2b^4$.

CASE II.

(82.) To extract any root of a polynomial, we have the following general

RULE.

I. Having arranged the polynomial according to the powers of some one of the letters, so that the highest power shall stand first, extract the required root of the first term, which will be the first term of the root sought.

II. Subtract the power of this first term of the root from the polynomial, and divide the first term of the remainder by the first term of the root involved to the next inferior power, and multiplied by the number denoting the root; the quotient will be the second term of the root.

III. Subtract the power of the terms already found from the polynomial; and using the same divisor, proceed as before.

This rule obviously verifies itself, since, whenever a new term is added to the root, the whole is raised to the given power, and the result is subtracted from the given polynomial. And when we thus find a power equal to the given polynomial, it is evident that the true root has been found.

1. What is the fifth root of

$$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5?$$

OPERATION. ROOT.

$$\begin{array}{rl}
 & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5(a+b. \\
 & \underline{a^5\qquad\qquad} \\
5a^4) & \quad 5a^4b \\
(a+b)^5= & \underline{a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.}
\end{array}$$

No remainder.

EXPLANATION.

We first found the fifth root of the first term a^5, to be a, which we placed to the right of the polynomial for the first term of the root. Raising a to the fifth power and subtracting it from the polynomial, we have $5a^4b$ for the first term of the remainder.

Since the number denoting the root is 5, we raise the first term of the root, a, to the fourth power, which thus becomes a^4; this multiplied by 5, the number denoting the root, gives $5a^4$ for our divisor.

Now, dividing $5a^4b$ by $5a^4$, we get b, which we write for the second term of the root.

Involving this root $a+b$ to the fifth power by actual multiplication, as was done in Ex. 2, Art. 77, we have

$$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5,$$

which subtracted from the given polynomial leaves no remainder, so that we know that $a+b$ is the true root.

2. What is the square root of $4x^4-16x^3+24x^2-16x+4$?

OPERATION. ROOT.

$$\begin{array}{rl}
 & 4x^4-16x^3+24x^2-16x+4(2x^2-4x+2. \\
(2x^2)^2= & \underline{4x^4\qquad\qquad} \\
4x^2) & \qquad -16x^3 \\
(2x^2-4x)^2= & \underline{4x^4-16x^3+16x^2} \\
4x^2) & \qquad\qquad\qquad 8x^2 \\
(2x^2-4x+2)^2= & \underline{4x^4-16x^3+24x^2-16x+4}
\end{array}$$

3. What is the square root of

$$16x^4+24x^3+89x^2+60x+100\,?$$

Ans. $4x^2+3x+10$.

4. What is the cube root of

$$a^6+3a^5-3a^4-11a^3+6a^2+12a-8\,?$$

Ans. a^2+a-2.

5. What is the sixth root of

$$a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6\,?$$

Ans. $a-b$.

6. What is the fourth root of

$$a^4-4a^3b+6a^2b^2-4ab^3+b^4\,?$$ *Ans.* $a-b$.

7. What is the seventh root of

$$m^7+7m^6n+21m^5n^2+35m^4n^3+35m^3n^4+21m^2n^5+7mn^6+n^7\,?$$

Ans. $m+n$.

(83.) If we carefully observe the law by which a polynomial is raised to the second power, we shall, by reversing the process, be enabled to deduce a rule for the extraction of the *square root* of a polynomial, which will be more simple than the above general rule, and of more interest, since the arithmetical rule is deduced from it.

By actual multiplication, we find

$$(a+b)^2=a^2+2ab+b^2,$$

$$(a+b+c)^2=a^2+2ab+b^2+2(a+b)c+c^2,$$

$$(a+b+c+d)^2$$
$$=a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2,$$

$$(a+b+c+d+e)^2$$
$$=\left\{\begin{array}{l}a^2+2ab+b^2+2(a+b)c+c^2\\ +2(a+b+c)d+d^2+2(a+b+c+d)e+e^2.\end{array}\right\}$$

&c. &c.

From the above we discover, that

(84.) ***The square of any polynomial is equal to the square of the first term, plus twice the first term into the second, plus the*** *square of the second; plus twice the sum of the first two into the third, plus the square of the third; plus twice the sum of the first three into the fourth, plus the square of the fourth; and so on.*

(85.) Hence, the square root of a polynomial can be found by the following

RULE.

I. After arranging the polynomial according to the powers of some one of the letters, take the root of the first term for the first term of the required root, and subtract its square from the polynomial.

II. Bring down the next two terms for a dividend. Divide it by twice the root just found, and add the quotient both to the root and to the divisor. Multiply the divisor thus increased, into the term last placed in the root, and subtract the product from the dividend.

III. Bring down two or three additional terms, and proceed as before.

EXAMPLES.

1. What is the square root of

$$a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2\,?$$

OPERATION. ROOT.

$$\begin{array}{ll}
 & a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2\,\lfloor a+b+c+d. \\
 & \underline{a^2\qquad\qquad} \\
 & \quad 2ab+b^2 \\
 & \quad \underline{2ab+b^2\qquad\qquad\qquad} \\
2(a+b)+c & \qquad\qquad 2(a+b)c+c^2 \\
 & \qquad\qquad \underline{2(a+b)c+c^2\qquad\qquad\qquad} \\
2(a+b+c)+d & \qquad\qquad\qquad\qquad\qquad 2(a+b+c)d+d^2 \\
 & \qquad\qquad\qquad\qquad\qquad \underline{2(a+b+c)d+d^2}
\end{array}$$

2. What is the square root of

$$4x^6+12x^5+5x^4-2x^3+7x^2-2x+1\,?$$

OPERATION. ROOT.

$$\begin{array}{ll}
 & 4x^6+12x^5+5x^4-2x^3+7x^2-2x+1\,(2x^3+3x^2-x+1. \\
 & 4x^6 \\ \hline
4x^3+3x^2 & \quad 12x^5+5x^4 \\
 & \quad 12x^5+9x^4 \\ \hline
4x^3+6x^2-x & \qquad -4x^4-2x^3+7x^2 \\
 & \qquad -4x^4-6x^3+\ x^2 \\ \hline
4x^3+6x^2-2x+1 & \qquad\qquad 4x^3+6x^2-2x+1 \\
 & \qquad\qquad 4x^3+6x^2-2x+1 \\ \hline
 & \qquad\qquad\qquad\qquad 0
\end{array}$$

Since the square of a minus quantity is the same as the square of the same quantity with the plus sign, it follows that the *square root* of any quantity, or of a polynomial, as well as of a monomial, is either plus or minus; that is, the root is affected with the double sign ±. The same is true of any even root.

Thus, if we change all the signs of the terms of the root just found, we shall have $-2x^3-3x^2+x-1$. If this is squared, we shall find that it will produce the primitive polynomial, whose root we have just extracted.

3. What is the square root of

$$x^4-2x^2y^2-2x^2+y^4+2y^2+1\,?$$

Ans. $\pm(x^2-y^2-1)$.

4. What is the square root of

$$9x^4y^4-30x^3y^3+25x^2y^2\,?$$

Ans. $\pm(3x^2y^2-5xy)$.

5. What is the square root of

$$a^2+2ab-2ac+b^2-2bc+c^2\,?$$

Ans. $\pm(a+b-c)$.

6. What is the square root of

$4m^2-36mn+81n^2$? *Ans.* $\pm(2m-9n)$.

7. What is the square root of

$$x^6+2x^5+x^4+2x^3+2x^2+1\,?$$

Ans. $\pm(x^3+x^2+1)$

8. What is the square root of

$$x^6+2x^5+3x^4+4x^3+3x^2+2x+1\,?$$

Ans. $\pm(x^3+x^2+x+1)$

9. What is the square root of

$$4x^2-12x^{\frac{3}{2}}+13x-6x^{\frac{1}{2}}+1\,?$$

Ans. $\pm(2x-3x^{\frac{1}{2}}+1)$.

10. What is the square root of

$$9x^2-12x^{\frac{5}{3}}+10x^{\frac{4}{3}}-10x+5x^{\frac{2}{3}}-2x^{\frac{1}{3}}+1\,?$$

Ans. $\pm(3x-2x^{\frac{2}{3}}+x^{\frac{1}{3}}-1)$.

11. What is the square root of

$$x^4-3x^3+3\tfrac{1}{4}x^2-1\tfrac{1}{2}x+\tfrac{1}{4}\,?$$

Ans. $\pm(x^2-1\tfrac{1}{2}x+\tfrac{1}{2})$.

12. What is the square root of

$$\tfrac{1}{9}a^4-\tfrac{4}{9}a^3b+\tfrac{8}{9}ab^3+\tfrac{4}{9}b^4\,?$$

Ans. $\pm(\tfrac{1}{3}a^2-\tfrac{2}{3}ab-\tfrac{2}{3}b^2)$.

IRRATIONAL OR SURD QUANTITIES.

(86.) **An Irrational Quantity, or Surd,** is a quantity affected with a fractional exponent or radical, without which it cannot be accurately expressed. Thus,

$\sqrt{3}$ is a surd, since the square root of 3 cannot be accurately found; also $8^{\frac{1}{2}}$, $4^{\frac{1}{3}}$, $\sqrt[3]{4}$, $\sqrt[4]{5}$, &c., are surd quantities.

REDUCTION OF SURDS.

CASE I.

(87.) To reduce a rational quantity to the form of a surd, we have this

RULE.

Raise the quantity to a power denoted by the root of the required surd; then the corresponding root of this power, expressed by means of a radical sign or fractional exponent, will express the quantity under the proposed form.

EXAMPLES.

1. Reduce $5a$ to the form of the cube root.

Raising $5a$ to the third power, we have

$$(5a)^3=125a^3;$$

Extracting the cube root, it becomes

$$5a=\sqrt[3]{125a^3}=(125a^3)^{\frac{1}{3}}.$$

2. Reduce $\dfrac{x^3}{a^3}$ to the form of the fifth root.

$$\textit{Ans.}\ \frac{x^3}{a^3}=\sqrt[5]{\frac{x^{15}}{a^{15}}}=\left(\frac{x^{15}}{a^{15}}\right)^{\frac{1}{5}}.$$

3. Reduce $\dfrac{\sqrt{a}}{y}$ to the form of the fourth root.

$$\textit{Ans.}\ \frac{\sqrt{a}}{y}=\left(\frac{a^2}{y^4}\right)^{\frac{1}{4}}.$$

4. Reduce $\dfrac{a^2}{b^3}$ to the form of the nth root.

$$\textit{Ans.}\ \frac{a^2}{b^3}=\left(\frac{a^{2n}}{b^{3n}}\right)^{\frac{1}{n}}.$$

5. Reduce $\dfrac{a}{b^{\frac{1}{2}}}$ to the form of the third root.

$$\textit{Ans.}\ \left(\frac{a^3}{b^{\frac{3}{2}}}\right)^{\frac{1}{3}}.$$

6. Reduce $\frac{x+y}{x-y}$ to the form of the square root.

Ans. $\left(\frac{(x+y)^2}{(x-y)^2}\right)^{\frac{1}{2}} = \left(\frac{x^2+2xy+y^2}{x^2-2xy+y^2}\right)^{\frac{1}{2}}$.

CASE II.

88. To reduce surds expressing different roots to equivalent ones expressing the same root.

RULE.

Reduce the different indices or exponents to common denominators; raise each quantity to a power denoted by the numerator of its respective index or exponent; and then take the root denoted by the common denominator.

EXAMPLES.

1. Reduce $\sqrt{3}$, $\sqrt[3]{4}$, and $\sqrt[4]{5}$, to surds expressing the same root.

Changing the radicals into fractional exponents, they become $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, which reduced to a common denominator, are $\frac{6}{12}$, $\frac{4}{12}$, $\frac{3}{12}$. Now, raising the quantities, 3, 4, and 5, to the powers denoted respectively by 6, 4, and 3, we find 3^6, 4^4, 5^3, or which is the same, 729, 256, 125. Taking the 12th root of these results, they become

$$(729)^{\frac{1}{12}},\ (256)^{\frac{1}{12}},\ (125)^{\frac{1}{12}}.$$

2. Reduce $a^{\frac{1}{2}}$ and $x^{\frac{2}{3}}$ to surds expressing the same root.

$$\textit{Ans.}\ \begin{cases} a^{\frac{1}{2}}=(a^3)^{\frac{1}{6}}, \\ x^{\frac{2}{3}}=(x^4)^{\frac{1}{6}}, \end{cases}$$

3. Reduce $x^{\frac{1}{3}}$, $y^{\frac{2}{3}}$, $m^{\frac{1}{2}}$, to surds expressing the same root.

$$\textit{Ans.}\ \begin{cases} x^{\frac{1}{3}}=(x^2)^{\frac{1}{6}}, \\ y^{\frac{2}{3}}=(y^4)^{\frac{1}{6}}, \\ m^{\frac{1}{2}}=(m^3)^{\frac{1}{6}}. \end{cases}$$

4. Reduce $\sqrt[3]{2}$, $\sqrt[4]{3}$, $\sqrt[5]{4}$, to surds expressing the same root.

$$Ans. \begin{cases} \sqrt[3]{2}=(2^{20})^{\frac{1}{60}} \\ \sqrt[4]{3}=(3^{15})^{\frac{1}{60}}, \\ \sqrt[5]{4}=(4^{12})^{\frac{1}{60}}. \end{cases}$$

5. Reduce $a^{\frac{1}{2}}$, $b^{\frac{1}{3}}$, $c^{\frac{1}{4}}$, to surds expressing the same root.

$$Ans. \begin{cases} (a^6)^{\frac{1}{12}}, \\ (b^4)^{\frac{1}{12}}, \\ (c^3)^{\frac{1}{12}}. \end{cases}$$

6. Reduce $(2x)^{\frac{1}{2}}$, $(3y)^{\frac{1}{3}}$, to the form of the sixth root.

$$Ans. \begin{cases} (8x^3)^{\frac{1}{6}}, \\ (9y^2)^{\frac{1}{6}}. \end{cases}$$

CASE III.

(89.) To reduce surds to their simplest form. Whenever a surd can be separated into two factors, one of which is a perfect power, it can be simplified by this

RULE.

Having separated the surd into two factors, one of which is a perfect power, take the root of the factor which is a perfect power, and multiply it by the surd of the other factor.

EXAMPLES.

1. Reduce $\sqrt{288}$ to its simplest form.

We can separate 288 into the factors 144×2, of which 144 is a perfect square, whose root is 12; therefore

$$\sqrt{288}=\sqrt{144\times 2}=\sqrt{144}\times\sqrt{2}=12\sqrt{2}.$$

2. Reduce $\sqrt[3]{x^3y-a^2x^3}$ to its simplest form.

Ans. $\sqrt[3]{x^3y-a^2x^3}=\sqrt[3]{x^3(y-a^2)}=x\sqrt[3]{y-a^2}$.

3. Reduce $\sqrt[5]{-32a^5b}$ to its simplest form.

Ans. $\sqrt[5]{-32a^5b}=-2a\sqrt[5]{b}$.

4. Reduce $(a^2y^{-4}x^6)^{\frac{1}{2}}$ to its simplest form.

Ans. $(a^2y^{-4}x^6)^{\frac{1}{2}}=ax^3y^{-2}=\frac{ax^3}{y^2}$.

5. Reduce $(m^2nx^5y^3)^{\frac{1}{2}}$ to its simplest form.

Ans. $mx^2y(nxy)^{\frac{1}{2}}$.

(90.) When a surd is in the form of a fraction, it may be simplified by the following

RULE.

Multiply both numerator and denominator by such a quantity as will render the denominator a perfect power.

EXAMPLES.

1. Reduce $\sqrt{\frac{8}{11}}$ to its simplest form.

Multiplying both numerator and denominator by 11, we have

$$\sqrt{\frac{8}{11}}=\sqrt{\frac{88}{121}}=\sqrt{\frac{4}{121}\times 22}=\frac{2}{11}\sqrt{22}.$$

2. Reduce $\sqrt[3]{\frac{ab^2}{x^2}}$ to its simplest form.

Ans. $\sqrt[3]{\frac{ab^2}{x^2}}=\sqrt[3]{\frac{ab^2x}{x^3}}=\frac{1}{x}\sqrt[3]{ab^2x}$.

3. Reduce $\left(\frac{a^4b^8}{xy}\right)^{\frac{1}{5}}$ to its simplest form.

Ans. $\left(\frac{a^4b^8}{xy}\right)^{\frac{1}{5}}=\frac{b}{xy}(a^4b^3x^4y^4)^{\frac{1}{5}}$.

4. Reduce $\left(\frac{a^{-1}b^{-2}}{x}\right)^{\frac{1}{3}}$ to its simplest form.

Ans. $\left(\frac{a^{-1}b^{-2}}{x}\right)^{\frac{1}{3}}=\left(\frac{1}{ab^2x}\right)^{\frac{1}{3}}=\frac{1}{abx}(a^2bx^2)^{\frac{1}{3}}$.

ADDITION AND SUBTRACTION OF SURDS.

RULE.

(91.) *Reduce the surds to their simplest form; then, if the surd part is the same in both, add or subtract the rational parts, and annex the common surd part to the result; but when the surd parts are different, they can only be added or subtracted by the aid of the signs + or —.*

EXAMPLES.

1. What is the sum of $\sqrt{54}$ and $\sqrt{24}$? Also, what is the difference of the same surds?

By reduction we have

$$\begin{array}{lll} & \sqrt{54}=\sqrt{9\times6} & =3\sqrt{6}. \\ & \underline{\sqrt{24}}=\sqrt{4\times6} & =\underline{2\sqrt{6}}. \\ \text{Therefore,} & \sqrt{54}+\sqrt{24} & =5\sqrt{6}. \\ \text{And,} & \sqrt{54}-\sqrt{24} & =\ \sqrt{6}. \end{array}$$

2. What is the sum and difference of $\sqrt[3]{a^4b}$ and $\sqrt[3]{ab^4}$?

$$\textit{Ans.} \begin{cases} \text{The sum } =(a+b)\sqrt[3]{ab}. \\ \text{The diff. } =(a-b)\sqrt[3]{ab}. \end{cases}$$

3. What is the sum of $(36x^2y)^{\frac{1}{2}}$ and $(25y)^{\frac{1}{2}}$?

Ans. $(6x+5)\sqrt{y}$.

4. What is the sum of $(8x)^{\frac{1}{3}}$, $(xy^6)^{\frac{1}{3}}$, and $(27x^4)^{\frac{1}{3}}$?

Ans. $(2+3x+y^2)\sqrt[3]{x}$.

5. What is the sum of $(ab^2x^6)^{\frac{1}{5}}$ and $(m^4y^{10})^{\frac{1}{5}}$?

Ans. $x(ab^2x)^{\frac{1}{5}}+y^2(m^4)^{\frac{1}{5}}$

6. What is the sum of $(16x)^{\frac{1}{2}}$ and $(9x)^{\frac{1}{2}}$?

Ans. $4x^{\frac{1}{2}}+3x^{\frac{1}{2}}=7x^{\frac{1}{2}}$.

7. What is the value of $(27x)^{\frac{1}{3}}-(8x)^{\frac{1}{3}}$?

Ans. $3x^{\frac{1}{3}}-2x^{\frac{1}{3}}=x^{\frac{1}{3}}$.

MULTIPLICATION AND DIVISION OF SURDS.

RULE.

(92.) *Reduce the surds to equivalent ones expressing the same root* (Case II., Art. 88), *then multiply or divide as required.*

EXAMPLES.

1. What is the product of $\sqrt{8}$ by $\sqrt[3]{16}$?

By Case II., we find

$$\sqrt{8}=(8^3)^{\frac{1}{6}}=(512)^{\frac{1}{6}},$$

$$\sqrt[3]{16}=(16^2)^{\frac{1}{6}}=(256)^{\frac{1}{6}}.$$

Therefore, $\sqrt{8}\times\sqrt[3]{16}=(512\times256)^{\frac{1}{6}}=4\sqrt[6]{32}.$

2. Divide $4\sqrt[6]{32}$ by $\sqrt[3]{16}$.

$$4\sqrt[6]{32}=2^2\times\sqrt[6]{2^5}=2^2\times2^{\frac{5}{6}}=2^{\frac{17}{6}}.$$

$$\sqrt[3]{16}=\sqrt[3]{2^4}=2^{\frac{4}{3}}=2^{\frac{8}{6}}.$$

$$2^{\frac{17}{6}}\div2^{\frac{8}{6}}=2^{\frac{9}{6}}=2^{\frac{3}{2}}=\textit{Ans.}\ \sqrt{8}.$$

3. What is the product of $4\sqrt[3]{ab}$ by $3\sqrt{by}$?

Ans. $12\sqrt[6]{a^2b^5y^3}$.

4. Divide $\sqrt{4a^3b^3}$ by $\sqrt[3]{2ab^2}$. *Ans.* $a(16ab^5)^{\frac{1}{6}}$.

5. Divide $\sqrt{a^2xy}$ by $\sqrt[3]{ax}$. *Ans.* $\sqrt[6]{a^4xy^3}$,

6. Divide $(ab)^{\frac{1}{5}}$ by $(ab)^{\frac{1}{10}}$. *Ans.* $(ab)^{\frac{1}{10}}$.

7. Multiply $(a+b)^{\frac{1}{5}}$ by $(x+y)^{\frac{1}{10}}$.

Ans. $[(a+b)^2\times(x+y)]^{\frac{1}{10}}$.

TO FIND MULTIPLIERS WHICH WILL CAUSE SURDS TO BECOME RATIONAL.

CASE I.

(93.) When the surd consists of but one term, we can proceed as follows:

Suppose the given surd is $x^{\frac{1}{m}}$; if we multiply this by $x^{\frac{m-1}{m}}$, by rule under Art. 92, we shall have $x^{\frac{1}{m}} \times x^{\frac{m-1}{m}} = x$, a rational quantity.

Hence, to cause a surd to become rational by multiplication, we have this

RULE.

Multiply the surd by the same quantity, having such an exponent as, when added to the exponent of the given surd, shall make a unit.

EXAMPLES.

1. How can the surd $x^{\frac{1}{3}}$ be made rational by multiplication?

In this example, $\frac{2}{3}$ added to the exponent $\frac{1}{3}$, gives 1; therefore we must multiply by $x^{\frac{2}{3}}$. Performing the operation, we have

$$x^{\frac{1}{3}} \times x^{\frac{2}{3}} = x.$$

2. Multiply $x^{\frac{3}{5}}$ so that it shall become rational.

Ans. $x^{\frac{3}{5}} \times x^{\frac{2}{5}} = x.$

3. Multiply $x^{-\frac{4}{7}}$ so that it shall become rational.

Ans. $x^{-\frac{4}{7}} \times x^{\frac{11}{7}} = x.$

4. What factor will make $a^{\frac{1}{2}}b^{\frac{1}{3}}$ rational? *Ans.* $a^{\frac{1}{2}}b^{\frac{2}{3}}$.

5. What factor will make $x^{\frac{1}{4}}y^{\frac{1}{5}}$ rational? *Ans.* $x^{\frac{3}{4}}y^{\frac{4}{5}}$,

CASE II.

(94.) When the surd consists of two terms, or is a binomial surd.

Suppose it is required to multiply $\sqrt{a}+\sqrt{b}$ so as to produce a rational product; we find that

$$(\sqrt{a}+\sqrt{b})\times(\sqrt{a}-\sqrt{b})=a-b.$$

Hence, to cause a binomial surd to become rational by multiplication, we have this

RULE.

Change the signs which connect the two terms of the binomial surd, from + to —, or from — to +, and this result, multiplied by the binomial surd, will give a rational product.

EXAMPLES.

1. Multiply $\sqrt{3}-\sqrt{2}$ so as to obtain a rational product.

Ans. $(\sqrt{3}-\sqrt{2})\times(\sqrt{3}+\sqrt{2})=3-2=1.$

2. Multiply $4+\sqrt{5}$ so that the result shall be rational.

Ans. $(4+\sqrt{5})\times(4-\sqrt{5})=11.$

3. How can $\sqrt{a+b}-\sqrt{a-b}$ be made rational by multiplication?

Ans. $(\sqrt{a+b}-\sqrt{a-b})\times(\sqrt{a+b}+\sqrt{a-b})=2b.$

4. How can $\sqrt{7}-1$ become rational by multiplication?

Ans. $(\sqrt{7}-1)\times(\sqrt{7}+1)=6.$

(95.) If the surd consists of three or more terms of the square root, connected by the signs plus and minus, it will require two or more successive multiplications.

EXAMPLES.

1. If it is required to make $\sqrt{5}-\sqrt{3}+\sqrt{2}$ rational by multiplication, we should first multiply by $\sqrt{5}+\sqrt{3}+\sqrt{2}$, by which means we obtain

$$
\begin{array}{l}
\sqrt{5}-\sqrt{3}+\sqrt{2} \\
\sqrt{5}+\sqrt{3}+\sqrt{2} \\
\hline
5-\sqrt{15}+\sqrt{10}-3+\sqrt{6} \\
\quad+\sqrt{15}+\sqrt{10}\qquad-\sqrt{6}+2 \\
\hline
5\qquad+2\sqrt{10}-3\qquad+2=2\sqrt{10}+4. \\
=2\times(\sqrt{10}+2).
\end{array}
$$

The irrational factor, $\sqrt{10}+2$, of this expression will become rational if we multiply it by $\sqrt{10}-2$. Hence, multiplying $2\sqrt{10}+4$ by $\sqrt{10}-2$, we have 12 for the rational product.

Since our original polynomial $\sqrt{5}-\sqrt{3}+\sqrt{2}$ was first multiplied by $\sqrt{5}+\sqrt{3}+\sqrt{2}$, and the result afterwards multiplied by $\sqrt{10}-2$, it follows that the product of $\sqrt{5}+\sqrt{3}+\sqrt{2}$ and $\sqrt{10}-2$ will give one single factor or multiplier, which would make $\sqrt{5}-\sqrt{3}+\sqrt{2}$ rational. Actually taking the product of $\sqrt{5}+\sqrt{3}+\sqrt{2}$ and $\sqrt{10}-2$, we have

$$
\begin{array}{l}
\sqrt{5}+\sqrt{3}+\sqrt{2} \\
\sqrt{10}-2 \\
\hline
\sqrt{50}+\sqrt{30}+\sqrt{20}-2\sqrt{5}-2\sqrt{3}-2\sqrt{2}.
\end{array}
$$

Since $\sqrt{50}=5\sqrt{2}$, and $\sqrt{20}=2\sqrt{5}$, the above becomes, $\sqrt{30}-2\sqrt{3}+3\sqrt{2}$. This result being used as a multiplier will at once make $\sqrt{5}-\sqrt{3}+\sqrt{2}$ rational. Actually multiplying, we have,

$$
\begin{array}{l}
\quad\sqrt{5}-\sqrt{3}+\sqrt{2} \\
\quad\sqrt{30}-2\sqrt{3}+3\sqrt{2} \\
\hline
\sqrt{150}-\sqrt{90}+\sqrt{60} \\
\qquad\qquad\qquad-2\sqrt{15}+6-2\sqrt{6} \\
\qquad\qquad\qquad\qquad\qquad+6-3\sqrt{6}+3\sqrt{10} \\
\hline
\sqrt{150}-\sqrt{90}+\sqrt{60}-2\sqrt{15}+12-5\sqrt{6}+3\sqrt{10}
\end{array}
$$

Observing that $\sqrt{150}=5\sqrt{6}$; $\sqrt{90}=3\sqrt{10}$; $\sqrt{60}=2\sqrt{15}$, we see that the above product is reduced to the rational part 12.

2. Multiply $2+\sqrt{3}-\sqrt{2}$ so that it shall become rational.

FIRST OPERATION.

$$\begin{array}{l}
2+\sqrt{3}-\sqrt{2}\\
2+\sqrt{3}+\sqrt{2}\\
\hline
4+2\sqrt{3}-2\sqrt{2}+3-\sqrt{6}\\
\quad+2\sqrt{3}+2\sqrt{2}\quad+\sqrt{6}-2\\
\hline
4+4\sqrt{3}\qquad+3\qquad-2=4\sqrt{3}+5.
\end{array}$$

SECOND OPERATION.

$$\begin{array}{l}
4\sqrt{3}+5\\
4\sqrt{3}-5\\
\hline
48\quad+20\sqrt{3}\\
\qquad-20\sqrt{3}-25\\
\hline
48\qquad\qquad-25=23.
\end{array}$$

3. Multiply $\sqrt{5}+\sqrt{2}-\sqrt{3}+1$ so that its product shall be rational.

FIRST OPERATION.

$$\begin{array}{l}
\sqrt{5}+\sqrt{2}-\sqrt{3}+1\\
\sqrt{5}-\sqrt{2}+\sqrt{3}+1\\
\hline
5+\sqrt{10}-\sqrt{15}+\sqrt{5}+\sqrt{6}-\sqrt{2}+\sqrt{3}\\
-2-\sqrt{10}+\sqrt{15}+\sqrt{5}+\sqrt{6}+\sqrt{2}-\sqrt{3}\\
-3\\
1\\
\hline
1\qquad\qquad+2\sqrt{5}+2\sqrt{6}.
\end{array}$$

SECOND OPERATION.

$$\begin{array}{l}
1+2\sqrt{5}+2\sqrt{6}\\
1-2\sqrt{5}+2\sqrt{6}\\
\hline
1+2\sqrt{5}+2\sqrt{6}-4\sqrt{30}\\
-20-2\sqrt{5}+2\sqrt{6}+4\sqrt{30}\\
24\\
\hline
5\qquad+4\sqrt{6}.
\end{array}$$

THIRD OPERATION.

$$\begin{array}{llll}
\multicolumn{2}{l}{4\sqrt{6}+5} & & \\
\multicolumn{2}{l}{4\sqrt{6}-5} & & \\
\hline
96 & +20\sqrt{6} & & \\
 & -20\sqrt{6} & -25 & \\
\hline
96 & & -25 & =71.
\end{array}$$

(96.) To reduce fractions having polynomial surds for a numerator, or denominator, or both, so that either the numerator or denominator may be free from radicals.

Suppose we wish to transform the fraction

$$\frac{1}{\sqrt{3}+\sqrt{2}+1}$$

into an equivalent fraction having a rational denominator.

It is evident that this transformation can be effected, provided we multiply both numerator and denominator by such a quantity as will cause the denominator to become free of radicals, so that the operation is reduced to the finding of a multiplier which will make

$$\sqrt{3}+\sqrt{2}+1 \text{ rational.}$$

We will first multiply by $-\sqrt{3}+\sqrt{2}+1$.

OPERATION.

$$\begin{array}{rrrrr}
 & \sqrt{3} & +\sqrt{2} & +1 & \\
- & \sqrt{3} & +\sqrt{2} & +1 & \\
\hline
- & 3 & -\sqrt{6} & -\sqrt{3} & +\sqrt{2} \\
+ & 2 & +\sqrt{6} & +\sqrt{3} & +\sqrt{2} \\
+ & 1 & & & \\
\hline
 & & & & 2\sqrt{2}.
\end{array}$$

Hence, if we multiply both the numerator and denominator of $\dfrac{1}{\sqrt{3}+\sqrt{2}+1}$ by $-\sqrt{3}+\sqrt{2}+1$, it will become $\dfrac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}}$

Again, multiplying both numerator and denominator of $\frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}}$ by $\sqrt{2}$, we finally have $\frac{\sqrt{2}-\sqrt{6}+2}{4}$.

The denominator is now rational.

(97.) Hence, to transform a fraction, having surds in its numerator or denominator, or both, into an equivalent fraction, in which the numerator or denominator may be free of surds, we have this

RULE.

Multiply the numerator and denominator by such a quantity as will cause the numerator or denominator, as the required case may be, to become rational.

EXAMPLES.

1. Reduce $\frac{5+\sqrt{3}}{4}$ to a fraction having a rational numerator.

Multiplying both numerator and denominator by $5-\sqrt{3}$, we have

$$\frac{5+\sqrt{3}}{4}=\frac{(5+\sqrt{3})\,(5-\sqrt{3})}{4(5-\sqrt{3})}=\frac{22}{20-4\sqrt{3}}$$

2. Reduce $\frac{\sqrt{7}-\sqrt{5}}{1+\sqrt{2}}$ to an equivalent fraction having a rational denominator. *Ans.* $\frac{\sqrt{14}-\sqrt{10}-\sqrt{7}+\sqrt{5}}{1}$.

3. Reduce $\frac{1}{\sqrt{3}-\sqrt{2}+1}$ to an equivalent fraction having a rational denominator. *Ans.* $\frac{2-\sqrt{2}+\sqrt{6}}{4}$.

4. Reduce $\frac{\sqrt{a}+\sqrt{x}}{\sqrt{b}+\sqrt{x}}$, first to a fraction having a rational de-

nominator, and then to a fraction having a rational numerator.

$$Ans. \begin{cases} \dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{b}+\sqrt{x}}=\dfrac{\sqrt{ab}-\sqrt{ax}+\sqrt{bx}-x}{b-x}. \\ \dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{b}+\sqrt{x}}=\dfrac{a-x}{\sqrt{ab}+\sqrt{ax}-\sqrt{bx}-x}. \end{cases}$$

IMAGINARY QUANTITIES.

(98.) We have already shown, that (see Note to the Rule under Art. 76) an even root of a negative quantity is impossible. Such expressions are called imaginary.

$$\left.\begin{array}{r} \sqrt{-a} \\ \sqrt[4]{-a} \\ \sqrt[6]{-a} \\ \sqrt[2m]{-a} \end{array}\right\} \text{are each imaginary quantities.}$$

Surd quantities, though their values cannot be accurately found, can, nevertheless, be approximately obtained; but imaginary quantities cannot have their values expressed by any means, either accurately or approximately. They must, therefore, be regarded merely as *symbolical expressions.*

(99.) We will confine ourselves to the imaginary expressions arising from taking the square root of a negative quantity.

The general form of imaginaries of this kind, is

$$\sqrt{-a}=\sqrt{a\times(-1)}=\sqrt{a}\,.\sqrt{-1}.$$

Replacing the surd quantity $\sqrt{a}$, by b, we have

$$\sqrt{-a}=b\sqrt{-1}.$$

So that all imaginary quantities arising from extracting the square root of a minus quantity are of the form

$$b\sqrt{-1}.$$

Where b is a real quantity, either rational or a surd.

(100.) If we put $\sqrt{-1}=c$, we shall always have

$$c^2=-1$$
$$c^3=-\sqrt{-1}$$
$$c^4=+1$$
$$c^5=+\sqrt{-1}.$$

And in general,

$$c^{4m}=+1$$
$$c^{4m+1}=+\sqrt{-1}$$
$$c^{4m+2}=-1$$
$$c^{4m+3}=-\sqrt{-1}.$$

m being any positive integer whatever.

(101.) From which we easily deduce the following principles:

1. $(+\sqrt{-a})\times(+\sqrt{-a})=-\sqrt{a^2}=-a.$
2. $(-\sqrt{-a})\times(-\sqrt{-a})=-\sqrt{a^2}=-a.$
3. $(+\sqrt{-a})\times(-\sqrt{-a})=+\sqrt{a^2}=+a.$
4. $(+\sqrt{-a})\times(+\sqrt{-b})=-\sqrt{ab}.$
5. $(-\sqrt{-a})\times(-\sqrt{-b})=-\sqrt{ab}.$
6. $(+\sqrt{-a})\times(-\sqrt{-b})=+\sqrt{ab}.$

The above is in accordance with the usual rules for the multiplication of algebraic quantities, and must be considered as a *definition* of this symbol, and of the method of using it, and not as a *demonstration* of its properties.

(102.) The student must not infer from what has been said, that imaginary quantities are useless. So far from being useless, they have lent their aid in the solution of questions which required the most refined and delicate analysis.

(103.) *Before closing this chapter, we will show the interpretation of the following symbols :*

$$\frac{0}{A},\quad \frac{A}{0},\quad \frac{0}{0}.$$

We know from the nature of multiplication, that 0 multiplied by a finite quantity, that is, 0 repeated a finite number of times, must still remain equal to 0 ; hence, we have this condition,

$$0\times A=0. \qquad (1)$$

Dividing both members of (1) by A, we find

$$0=\frac{0}{A}. \qquad (2)$$

Therefore the symbol $\frac{0}{A}$, will always be equal to 0, as long as A is a finite quantity

(104.) Since the quotient arising from dividing one quantity by another, becomes greater in proportion as the divisor is diminished, it follows that when the divisor becomes less than any assignable quantity, then the quotient will exceed any assignable quantity. Hence, it is usual for mathematicians to say, that $\frac{A}{0}$ is the representation of an infinite quantity. The symbol employed to represent *infinity* is ∞, so that we have

$$\frac{A}{0}=\infty. \qquad (3)$$

(105.) Dividing both members of (1) by 0, we find

$$A=\frac{0}{0} \qquad (4)$$

This being true for all values of A, shows that $\frac{0}{0}$ is the symbol of an *indeterminate quantity.*

To illustrate this last symbol, we will take several

EXAMPLES.

1. What is the value of the fraction $\frac{x^2-a^2}{bx-ab}$, when $x=a$?

Substituting a for x, our fraction will become

$$\frac{x^2-a^2}{bx-ab}=\frac{a^2-a^2}{ab-ab}=\frac{0}{0}=\text{ an indeterminate quantity.}$$

If, before substituting a for x, we divide both numerator and denominator of the given fraction by $x-a$ (Art. 42), we find

$$\frac{x^2-a^2}{bx-ab}=\frac{x+a}{b}.$$

Now, substituting a for x, in this reduced form, we find

$$\frac{x+a}{b}=\frac{a+a}{b}=\frac{2a}{b}.$$

Therefore, $\frac{2a}{b}$ is the true value of $\frac{x^2-a^2}{bx-ab}$, when $x=a$.

2. What is the value of $\frac{x^2-ax}{x^2-2ax+a^2}$, when $x=a$?

Substituting a for x, we find

$$\frac{x^2-ax}{x^2-2ax+a^2}=\frac{a^2-a^2}{a^2-2a^2+a^2}=\frac{0}{0}.$$

If we reduce this fraction by dividing both numerator and denominator by $x-a$, we find

$$\frac{x^2-ax}{x^2-2ax+a^2}=\frac{x}{x-a}.$$

Now, writing a for x in this reduced form, we find

$$\frac{x}{x-a}=\frac{a}{a-a}=\frac{a}{0}=\infty. \quad \text{(Art. 104.)}$$

3. What is the value of $\dfrac{x^3-3ax^2+3a^2x-a^3}{bx-ab}$, when $x=a$?

When a is substituted for x, we have

$$\frac{x^3-3ax^2+3a^2x-a^3}{bx-ab}=\frac{a^3-3a^3+3a^3-a^3}{ab-ab}=\frac{0}{0}.$$

Reducing, by dividing numerator and denominator by $x-a$, we find

$$\frac{x^3-3ax^2+3a^2x-a^3}{bx-ab}=\frac{x^2-2ax+a^2}{b}.$$

Writing a for x, we have

$$\frac{x^2-2ax+a^2}{b}=\frac{a^2-2a^2+a^2}{b}=\frac{0}{b}=0. \quad \text{(Art. 103.)}$$

(106.) From the above, we conclude that whenever an algebraic fraction is reduced to the form $\frac{0}{0}$ there exists a factor common to both numerator and denominator.

If we reduce the fraction, by dividing both numerator and denominator by this common factor, it will become of one of the following forms:

$$\frac{A}{B}=\textit{a finite quantity.}$$

$$\frac{0}{A}=0=\textit{ no value.}$$

$$\frac{A}{0}=\infty=\textit{an infinite quantity.}$$

CHAPTER V.

QUADRATIC EQUATIONS.

(107.) We have already (Art. 53) defined a *quadratic equation* to be, an equation in which the unknown quantity does not exceed the second degree.

The most general form of a quadratic equation of one unknown quantity is

$$ax^2+bx=c. \qquad (1)$$

Dividing each of the terms of (1) by a (Axiom IV.), we find

$$x^2+\frac{b}{a}x=\frac{c}{a}, \qquad (2)$$

where, if we assume $A=\frac{b}{a}$, and $B=\frac{c}{a}$, we shall have

$$x^2+Ax=B. \qquad (3)$$

Equation (3) is as general a form for quadratics as equation (1).

In (3), A and B can have any values either positive or negative.

(108.) When $A=0$, equation (3) will become

$$x^2=B, \qquad (4)$$

which is called an *incomplete quadratic equation*, since one of the terms in the general forms (1) and (3) is wanting.

(109.) When $B=0$, equation (3) will become

$$x^2+Ax=0,$$

which divided by x is reduced to

$$x+A=0,$$

which is no longer a quadratic equation, but a simple equation.

(110.) If $A=0$ and $B=0$ at the same time, equation (3) will become

$$x^2=0,$$

which can only be satisfied by taking $x=0$.

INCOMPLETE QUADRATIC EQUATIONS.

(111.) We have just seen that the general form of an incomplete quadratic equation is

$$x^2=B. \qquad (1)$$

If we extract the square root of both members of this equation, we shall (Art. 79) have

$$x=\pm\sqrt{B}. \qquad (a)$$

Equation (a) may be regarded as a general solution of incomplete quadratic equations.

(112.) To find the value of the unknown, when the equation which involves it leads to an incomplete quadratic equation, we have this

RULE.

I. Clear the equation of fractions by the same rule as for simple equations.

II. Then transpose and unite the like terms, if necessary, observing the rule under Art. 60, *and we shall thus obtain, after dividing by the coefficient of* x^2, *an equation of the form of* $x^2=B$. *Extracting the square root of both members, we shall find* $x=\pm\sqrt{B}$.

NOTE.—If B is negative, the value $\pm\sqrt{B}$, of x, will be *imaginary.*

EXAMPLES.

1. Given $\frac{x^2+2}{19}+7=9$, to find the values of x.

This, when cleared of fractions, by multiplying by 19, becomes

$$x^2+2+133=171.$$

Transposing and uniting terms, we find $x^2=36$. If we compare this with our general form, we shall see that $B=36$. Extracting the square root, we have $x=\pm6$, or as it may be better expressed, $x=6$, or $x=-6$.

We must be careful to interpret the double sign ±, correctly, the meaning of which is, that the quantity before which it is placed may be either plus, or it may be minus. It does not mean that the quantity can be both plus and minus at the same time.

2. Given $\frac{3}{14x^2}+\frac{1}{2}=\frac{346}{686}$, to find the values of x.

This cleared of fractions, becomes

$$147+343x^2=346x^2.$$

Transposing and uniting terms, $3x^2=147$.

Dividing by 3, $x^2=49$.

Extracting the square root, we find $x=\pm7$.

3. Given $x^2-\frac{25x^2}{36}=44$, to find the values of x.

Ans. $x=\pm12$.

4. Given $8+5x^2=\frac{x^2}{5}+4x^2+28$, to find the values of x.

Ans. $x=\pm5$.

5. Given $2+\frac{x^2}{3}-7=\frac{x^2}{9}+13$, to find the values of x.

Ans. $x=\pm9$.

6. Find the values of x from the equation

$$2x^2-35=80+\frac{15-x^2}{2}.$$ *Ans.* $x=\pm7$.

7. Find the values of x from the equation

$$6+\frac{3x(x+1)}{5}=7+\frac{3x-2}{5}.$$ *Ans.* $x=\pm1$.

8. Find the values of x from the equation

$$\frac{x^2+7x+17}{35}=\frac{3+x}{5}. \qquad \textit{Ans.}\ x=\pm 2.$$

9. Find the values of x from the equation

$$\frac{x(x^2+1)}{20}=\frac{x}{2}. \qquad \textit{Ans.}\ x=\pm 3.$$

10. Find the values of x from the equation

$$\frac{x(x+3)}{9}-\frac{x+1}{3}=\frac{4}{9}+1. \qquad \textit{Ans.}\ x=\pm 4.$$

11. Given $\frac{945-171x^2}{43}=\frac{37-x^2}{2}$, to find the values of x.

Ans. $x=\pm 1$.

12. Given $\frac{x^2-8x+80}{2}=3x^2-4x$, to find the values of x.

Ans. $x=\pm 4$.

13. Given $\frac{x^2+x}{4}=\frac{4x^2+5x-1}{20}$, to find the values of x.

Ans. $x=\pm\sqrt{-1}$.

Both values being imaginary.

14. Given $\frac{3x^2+35x}{15}+30=\frac{x^2+4x}{3}+x$, to find the values of x.

Ans. $x=\pm 15$.

15. Given $\frac{x^2}{5}+\frac{7x}{3}-\frac{8}{15}=\frac{x^2+x}{3}+2x$, to find x.

Ans. $x=\pm\sqrt{-4}=\pm 2\sqrt{-1}$,

which are imaginary values.

16. Given $\frac{x^2}{5}+\frac{2}{15}=\frac{x^2+x}{3}-\frac{x}{3}$, to find x. *Ans.* $x=\pm 1$.

(113.) If an equation involving one unknown quantity can be reduced to the form $x^n=N$, the value of x can be found by simply extracting the nth root of both members; thus,

$$x=\sqrt[n]{N}.$$

(114.) Where it must be observed (Art. 79), that when n is an *even* number, the value of x will be either plus or minus for all positive values of N; but for negative values of N the value of x will be impossible. When n is an *odd* number, the value of x will have the same sign that N has.

(115.) If the equation can be reduced to the form $x^{\frac{1}{m}}=N$, then x can be found by raising both members to the mth power, thus:

$$x=N^m.$$

Where x will be positive for all values of N, provided m is an *even* number; but when m is an *odd* number, then x will have the same sign as N.

(116.) Finally, when the equation can be reduced to the form

$$x^{\frac{n}{m}}=N,$$

we must first involve both members to the mth power, and then extract the nth root, or else we may first extract the nth root, and then involve to the mth power (Art. 81).

Thus, $$x=N^{\frac{m}{n}}.$$

EXAMPLES.

1. Given $\dfrac{\sqrt{x}+28}{\sqrt{x}+4}=\dfrac{\sqrt{x}+38}{\sqrt{x}+6}$, to find x.

This, when cleared of fractions, becomes

$$x+34\sqrt{x}+168=x+42\sqrt{x}+152.$$

Transposing and uniting terms, we have

$$8\sqrt{x}=16.$$

Dividing by 8, $\sqrt{x}=2$.

Raising to the second power, $x=4$.

2. Given $\sqrt{x}+\sqrt{a+x}=\dfrac{2a}{\sqrt{a+x}}$, to find x.

This equation, when cleared of fractions, by multiplying by $\sqrt{a+x}$, becomes

$$\sqrt{ax+x^2}+a+x=2a,$$

or, $$\sqrt{ax+x^2}=a-x.$$

Squaring both members,

$$ax+x^2=a^2-2ax+x^2,$$

or, $$3ax=a^2,$$

or, $$x=\frac{a}{3}.$$

3. Given $3+x^{\frac{2}{3}}=7$, to find the values of x. *Ans.* $x=\pm 8$.

4. Given $(y^n-b)^{\frac{1}{2}}=a-d$, to find the values of y.

Ans. $y=\left\{(a-d)^2+b\right\}^{\frac{1}{n}}$.

5. Given $\sqrt{x-32}=16-\sqrt{x}$, to find the value of x.

Ans. $x=81$.

6. Given $(x+a)^{\frac{1}{2}}=\dfrac{a+b}{(x-a)^{\frac{1}{2}}}$, to find the values of x.

Ans. $x=\pm(2a^2+2ab+b^2)^{\frac{1}{2}}$.

7. Given $\dfrac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}-\sqrt{x-1}}=\dfrac{4}{x-1}$, to find the values of x.

If we multiply the numerator and denominator of the left-hand member by the numerator, it will become

$$(\sqrt{x}+\sqrt{x-1})^2=\frac{4}{x-1}.$$

Extracting the square root, we find

$$\sqrt{x}+\sqrt{x-1}=\frac{\pm 2}{\sqrt{x-1}}.$$

This readily gives $x=\frac{9}{5}$, or $x=-\frac{1}{3}$.

8. Given $(x^3-15)^{\frac{1}{2}}=7$, to find x. *Ans.* $x=4$.

9. Given $(x^4-7)^{\frac{1}{2}}=3$, to find the values of x. *Ans.* $x=\pm 2$.

10. Given $(x^{\frac{3}{4}}-1)^2=49$, to find x. *Ans.* $x=16$.

11. Given $\frac{\sqrt{x}+1}{\sqrt{x}+5}=\frac{\sqrt{x}-3}{\sqrt{x}+4}$, to find x. *Ans.* $x=\frac{361}{9}$.

COMPLETE QUADRATIC EQUATIONS.

(117.) We have already seen, that

$$ax^2+bx=c, \qquad \text{(A)}$$

is the most general form of a quadratic equation, where

$a =$ the coefficient of the first term,
$b =$ the coefficient of the second term,
$c =$ the term independent of x.

Since x enters into this equation in the second power, we must extract the square root before we can find its value. We cannot extract the square root of the left-hand member as it now is, unless a, the coefficient of x^2, is a square number. If we multiply both members of this general form by a, it will become

$$a^2x^2+abx=ac. \qquad (1)$$

We will now attempt to extract the square root of the left-hand member by the Rule under Art. 84.

$$\begin{array}{lll} & a^2x^2+abx & \left(ax+\frac{b}{2}\right. . \\ & \underline{a^2x^2} & \\ 2ax+\frac{b}{2} & abx & \\ & \underline{abx+\frac{b^2}{4}} & \end{array}$$

From the operation, thus far extended, we see that the root of the left-hand member would be $ax+\frac{b}{2}$, provided the left-hand member had an additional term of $\frac{b^2}{4}$. We will therefore add to both members of (1), $\frac{b^2}{4}$. It will thus become

$$a^2x^2+abx+\frac{b^2}{4}=\frac{b^2}{4}+ac.$$

The left-hand member is now a perfect square. If we multiply it by any square number, the result will still remain a perfect square. Multiplying by the square number 4, the fractions will disappear, and the resulting equation will become

$$4a^2x^2+4abx+b^2=b^2+4ac. \qquad (2)$$

Now extracting the square root of both members of (2), we shall obtain

$$2ax+b=\pm\sqrt{b^2+4ac}. \qquad (3)$$

By transposition, (3) gives

$$2ax=-b\pm\sqrt{b^2+4ac}. \qquad (4)$$

Dividing (4) by $2a$, we get

$$x=\frac{-b\pm\sqrt{b^2+4ac}}{2a}. \qquad (B)$$

Formula (B) may be regarded as a general solution of all complete quadratic equations.

If we translate it into common language, we shall obtain a rule for solving complete quadratics.

Hence, to find the value of an unknown quantity, when given by a quadratic equation, we have this

RULE.

Having reduced the equation to the general form $ax^2+bx=c$, *we can find* x, *by taking the coefficient of the second term with* *its sign changed, plus or minus the square root of the square of the coefficient of the second term increased by four times the coefficient of the first term into the term independent of x; and dividing the whole by twice the coefficient of the first term.*

EXAMPLES.

1. Given $4x-\frac{36-x}{x}=46$, to find the values of x.

This, when cleared of fractions, becomes

$$4x^2-36+x=46x$$

Transposing and uniting terms, we have

$$4x^2-45x=36.$$

This compared with the general form

$$ax^2+bx=c,$$

gives $\quad a=4;\ b=-45;\ c=36.$

The square of the coefficient of the second term is

$$(-45)^2=2025.$$

Four times the coefficient of the first term into the term independent of x, is

$$4\times4\times36=576.$$

Therefore, taking the square root of the square of the coefficient of the second term increased by four times the coefficient of the first term into the term independent of x, we get

$$\pm\sqrt{2025+576}=\pm\sqrt{2601}=\pm51.$$

This added to the coefficient of the second term with the sign changed, gives

$$45\pm51.$$

which must be divided by twice the coefficient of the first term. Hence,

$$x=\frac{45\pm51}{8}.$$

If we take the upper sign, we get

$$x=\frac{45+51}{8}=12.$$

If we take the lower sign, we find

$$x=\frac{45-51}{8}=-\tfrac{3}{4}.$$

Therefore, $x=12$, or $-\tfrac{3}{4}$.

Either of which values of x will verify the equation

2. Given $\dfrac{3x-4}{x-4}=9-\dfrac{x-2}{2}$, to find the values of x.

This, when reduced to the general form, becomes

$$x^2-18x=-72.$$

Squaring 18, we get

$$(18)^2=324.$$

Four times the first coefficient multiplied into -72, gives

$$4\times-72=-288,$$

which added to 324, gives 36, the square root of which is ±6.

Therefore, $x=\dfrac{18\pm6}{2}=12$, or 6.

3. Given $\sqrt{3x-5}=\dfrac{\sqrt{7x^2+36x}}{x}$, to find the values of x.

Squaring both members, we have

$$3x-5=\frac{7x^2+36x}{x^2}.$$

This, cleared of fractions, becomes

$$3x^2-5x=7x+36.$$

Transposing and uniting terms, we have

$$3x^2-12x=36.$$

This, divided by 3, gives

$$x^2-4x=12.$$

Therefore, $x=\frac{4\pm\sqrt{(4)^2+4\times 12}}{2}=\frac{4\pm 8}{2}=6$, or -2.

4. Given $\frac{3}{x^2-3x}+\frac{3}{x^2+4x}=\frac{27}{8x}$, to find the values of x.

This, by reduction, becomes

$$9x^2-7x=116.$$

Therefore, $x=\frac{7\pm\sqrt{7^2+4\times 9\times 116}}{18}=\frac{7\pm 65}{18}=4$, or $-3\frac{2}{9}$.

5. Given $\frac{x^2+12}{2}+\frac{x}{2}=4x$, to find the values of x.

This reduced, becomes

$$x^2-7x=-12.$$

Therefore, $x=\frac{7\pm\sqrt{7^2+4\times\overline{-12}}}{2}=\frac{7\pm 1}{2}=4$, or 3.

(118.) An equation of the form

$$ax^{2n}+bx^n=c, \qquad (A)$$

can be solved by the above rule, which indeed will agree with the form under consideration in the particular case of $n=1$.

If, in the above equation, we write y for x^n, and consequently y^2 for x^{2n}, it will become

$$ay^2+by=c,$$

which is precisely of the form of (A), Art. 117. Consequently,

$$y=\frac{-b\pm\sqrt{b^2+4ac}}{2a}$$

Resubstituting x^n for y, we have

$$x^n=\frac{-b\pm\sqrt{b^2+4ac}}{2a}.$$

And, $$x=\left\{\frac{-b\pm\sqrt{b^2+4ac}}{2a}\right\}^{\frac{1}{n}}. \qquad \text{(B)}$$

This value of x must hold for all values of the constants n, a, b, and c, whether positive or negative, integral or fractional.

1. Given $x^4+ax^2=b$, to find x.

This becomes $y^2+ay=b$, when for x^2 we write y.

Therefore, $$y=\frac{-a\pm\sqrt{a^2+4b}}{2}=x^2.$$

Hence, $$x=\pm\left\{\frac{-a\pm\sqrt{a^2+4b}}{2}\right\}^{\frac{1}{2}}.$$

2. Given $3x^{2n}-2x^n=8$, to find x.

$$x^n=\frac{2\pm10}{6}=2.$$

Therefore, $$x=\sqrt[n]{2}.$$

3. Given $2(1+x-x^2)-\sqrt{1+x-x^2}=-\frac{1}{9}$, to find x.

If, for $1+x-x^2$, we put y^2, our equation will become

$$2y^2-y=-\tfrac{1}{9},$$

or, $$18y^2-9y=-1.$$

Therefore, $$y=\frac{9\pm3}{36}=\tfrac{1}{3}, \text{ or } \tfrac{1}{6}.$$

Hence, $$y^2=\tfrac{1}{9}, \text{ or } \tfrac{1}{36}.$$

Resubstituting $1+x-x^2$, for y^2, we have, when we take the first value of y^2,

$$1+x-x^2=\tfrac{1}{9},$$

or, $$9x^2-9x=8.$$

Therefore, $x=\dfrac{9\pm3\sqrt{41}}{18}=\frac{1}{2}+\frac{1}{6}\sqrt{41}$, or $\frac{1}{2}-\frac{1}{6}\sqrt{41}$.

When we take the other value of y^2, we have

$$1+x-x^2=\tfrac{1}{36},$$

or, $$36x^2-36x=35.$$

Therefore, $x=\dfrac{36\pm24\sqrt{11}}{72}=\frac{1}{2}+\frac{1}{3}\sqrt{11}$, or $\frac{1}{2}-\frac{1}{3}\sqrt{11}$.

Collecting these four values of x, we find

$$x=\tfrac{1}{2}+\tfrac{1}{6}\sqrt{41},$$
$$x=\tfrac{1}{2}-\tfrac{1}{6}\sqrt{41},$$
$$x=\tfrac{1}{2}+\tfrac{1}{3}\sqrt{11},$$
$$x=\tfrac{1}{2}-\tfrac{1}{3}\sqrt{11}.$$

4. Given $x^4-25x^2=-144$, to find the four values of x. Let $x^2=y$, and the above equation will become

$$y^2-25y=-144.$$

Hence, $$y=16, \text{ or } y=9.$$

Consequently, $x=+4$, or -4; or else $x=+3$, or -3.

5. Given $x^4-7x^2=8$, to find the four values of x. Assume $x^2=y$, and we find

$$y^2-7y=8.$$

This solved, gives

$$y=8, \text{ or } y=-1.$$

When $y=8$, we find

$$x=\pm\sqrt{8}.$$

But when $y=-1$, we find

$$x=\pm\sqrt{-1}.$$

These two last values are *impossible.*

6. Given $2x-7\sqrt{x}=99$, to find x.

If we let $\sqrt{x}=y$, we shall have

$$2y^2-7y=99.$$

Hence, $y=9$, or $y=-\frac{11}{2}$.

If we take 9 for the value of y, we find

$$x=81.$$

But if we take $-\frac{11}{2}$ for the value of y, we find $x=\frac{121}{4}$.

7. Given $2x^2+\sqrt{2x^2+1}=11$, to find the four values of x.

Adding 1 to both members of the equation, we have

$$2x^2+1+\sqrt{2x^2+1}=12.$$

Assume $2x^2+1=y^2$,

and we obtain $y^2+y=12$.

This gives $y=3$, or $y=-4$.

Hence we have these two equations,

$$2x^2+1=9,$$
$$2x^2+1=16.$$

The first of these gives

$$x=2, \text{ or } x=-2.$$

The second gives

$$x=\tfrac{1}{2}\sqrt{30}, \text{ or } x=-\tfrac{1}{2}\sqrt{30}.$$

8. Given $x^6-8x^3=513$, to find one of the values of x.

Ans. $x=3$.

9. Given $x^4+4x^2=12$, to find two values of x.

Ans. $x=\pm\sqrt{2}$.

10. Given $x^{\frac{1}{3}}-x^{\frac{1}{6}}=2$, to find the values of x. *Ans.* $x=1$, or 64.

11. Given $x^{\frac{4}{3}}+x^{\frac{2}{3}}=20$, to find the values of x.

Ans. $x=\pm8$, or $\pm5\sqrt{-5}$.

(119.) When in the general form for quadratics $a=1$, the Rule under Art. 117 is susceptible of considerable modification.

If we substitute 1 for a in formulas (A) and (B), Art. 117, they will become

$$x^2+bx=c. \qquad \text{(C)}$$

$$x=\frac{-b\pm\sqrt{b^2+4c}}{2}=-\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2+c}. \qquad \text{(D)}$$

Therefore, when the coefficient of x^2 is a unit, the value of x can be found from formula (D).

All quadratic equations may be put under the form of (C), by dividing all the terms by the coefficient of x^2, so that the expression of (D) for x must be as general as that of (B), Art. 117.

Hence, for the solution of quadratic equations, we have this second

RULE.

Having reduced the equation to the form $x^2+bx=c$, we can find x, by taking half the coefficient of the second term, with its sign changed, plus or minus the square root of the square of the half of the coefficient of the second term increased by the term independent of x.

EXAMPLES.

1. Given $x^2-10x=-24$, to find x.

In this example, half the coefficient of the second term is 5, which squared and added to -24, the term independent of x, is 1. Extracting the square root of 1, we have ± 1.

Therefore, $\qquad x=5\pm 1=6$, or 4.

2. Given $\dfrac{x}{x+60}=\dfrac{7}{3x-5}$, to find x.

This, cleared of fractions, becomes

$$3x^2-5x=7x+420.$$

Transposing and uniting terms, we have

$$3x^2-12x=420.$$

Dividing by 3, we have

$$x^2-4x=140.$$

Therefore, $x=2\pm12=14$, or -10.

3. Given $\frac{x+12}{x}+\frac{x}{x+12}=\frac{26}{5}$, to find x.

Ans. $x=3$, or -15.

4. Given $3x^6+42x^3=3321$, to find x.

Ans. $x=3$, or $(-41)^{\frac{1}{3}}$.

5. Given $x(x-1)-\frac{x^2}{5}=(5-x)\left(1+\frac{x}{5}\right)+1$, to find x.

Ans. $x=-2$, or $+3$.

6. Given $\frac{x^2-x}{2}+3x=8-\frac{x}{2}$, to find x. *Ans.* $x=2$, or -8.

7. Given $x^2-8x=-17$, to find the values of x.

Ans. $x=4+\sqrt{-1}$, or $4-\sqrt{-1}$.

8. Given $x^2-6x=16$, to find the values of x.

Ans. $x=8$, or -2.

9. Given $6x^2-5x=-1$, to find the values of x.

Ans. $x=\frac{1}{2}$, or $\frac{1}{3}$.

10. Given $\frac{x^2}{a^2}-\frac{2x}{b}=-\frac{a^2}{b^2}$, to find the value of x.

Ans. $x=\frac{a^2}{b}$.

11. Given $2x^2-5x=-2$, to find the values of x.

Ans. $x=2$, or $\frac{1}{2}$.

12. Given $x^2-(a+b)x=-ab$, to find the values of x.

Ans. $x=a$, or b.

13. Given $ax^2+(a^2+1)x=-a$, to find the values of x.

Ans. $x=-a$, or $-\frac{1}{a}$.

14. Given $10x^2-101x=-10$, to find the values of x.

Ans. $x=10$, or $\frac{1}{10}$.

(120.) EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES, WHICH INVOLVE IN THEIR SOLUTION QUADRATIC EQUATIONS.

EXAMPLES.

1. Given $\left\{\begin{aligned} x+y&=10, & (1)\\ xy&=16, & (2)\end{aligned}\right\}$, to find x and y.

The square of (1) is

$$x^2+2xy+y^2=100. \qquad (3)$$

Subtracting 4 times the second from the (3), we get

$$x^2-2xy+y^2=36. \qquad (4)$$

The square root of (4) is

$$x-y=\pm 6. \qquad (5)$$

Half the sum of (1) and (5), gives

$$x=8, \text{ or } 2.$$

Half the difference of (1) and (5), gives

$$y=2, \text{ or } 8.$$

In this example, it will be seen that if x and y are interchanged, the two equations will in no respect be altered. Hence, the values of x and y, when found, ought to be interchangeable one with the other: and this we see is really the case; since when x is 8 then y is 2, but when x is 2 then y is 8.

2. Given $\left\{\begin{aligned} x+y&=a, & (1)\\ x^2+y^2&=b, & (2)\end{aligned}\right\}$, to find x and y.

Squaring (1), we have

$$x^2+2xy+y^2=a^2. \qquad (3)$$

Subtracting (2) from (3), we get

$$2xy=a^2-b. \qquad (4)$$

Subtracting (4) from (2), we find

$$x^2-2xy+y^2=2b-a^2. \qquad (5)$$

Extracting the square root of (5), we get

$$x-y=\pm\sqrt{2b-a^2}. \qquad (6)$$

Taking half the sum of (1) and (6), we get

$$x=\frac{a}{2}\pm\frac{1}{2}\sqrt{2b-a^2}. \qquad (7)$$

Subtracting (7) from (1), we find

$$y=\frac{a}{2}\mp\frac{1}{2}\sqrt{2b-a^2}.$$

In this example, as in the preceding one, we see that the values of x and y are interchangeable.

When $\qquad x=\frac{a}{2}+\frac{1}{2}(2b-a^2)^{\frac{1}{2}},$

then $\qquad y=\frac{a}{2}-\frac{1}{2}(2b-a^2)^{\frac{1}{2}},$

but when $\qquad x=\frac{a}{2}-\frac{1}{2}(2b-a^2)^{\frac{1}{2}},$

then will $\qquad y=\frac{a}{2}+\frac{1}{2}(2b-a^2)^{\frac{1}{2}}.$

3. Given $\left\{\begin{array}{ll} x(y+z)=a, & (1) \\ y(z+x)=b, & (2) \\ z(x+y)=c, & (3) \end{array}\right\}$ to find x, y, and z.

Before proceeding to the solution of these equations, we will remark, that they are symmetrical, and consequently all the derived equations will either contain all the letters similarly combined, or else they will appear in systems of three equations each, which can be deduced from each other by simply permuting (Art. 73).

If we take the sum of (1), (2), and (3), after expanding them, we shall have

$$2xy+2yz+2zx=a+b+c. \qquad (4)$$

In this equation all the letters enter symmetrically; therefore it will not give rise to any new equation by permutation.

If we subtract twice (3) from (4), we get

$$2xy=a+b-c. \qquad (5)$$

By permutation, we derive from (5) these two equations,

$$2yz=b+c-a, \qquad (6)$$

$$2zx=c+a-b. \qquad (7)$$

Equations (5), (6), and (7) readily give

$$xy=\frac{a+b-c}{2}. \qquad (8)$$

$$yz=\frac{b+c-a}{2}. \qquad (9)$$

$$zx=\frac{c+a-b}{2}. \qquad (10)$$

Taking the continued product of (8), (9), and (10), we shall have

$$x^2y^2z^2=\left(\frac{a+b-c}{2}\right)\left(\frac{b+c-a}{2}\right)\left(\frac{c+a-b}{2}\right). \qquad (11)$$

This equation containing all the letters symmetrically combined, can give no new condition by permutation.

Dividing (11) by the square of (9), we shall have

$$x^2=\frac{(a+b-c)(c+a-b)}{2(b+c-a)}. \qquad (12)$$

By permuting, we derive from (12) these two equations:

$$y^2=\frac{(b+c-a)\,(a+b-c)}{2(c+a-b)}. \qquad (13)$$

$$z^2=\frac{(c+a-b)\,(b+c-a)}{2(a+b-c)}. \qquad (14)$$

Taking the square roots of (12), (13), and (14), we find

$$x=\pm\left\{\frac{(a+b-c)\,(c+a-b)}{2(b+c-a)}\right\}^{\frac{1}{2}}. \qquad (15)$$

$$y=\pm\left\{\frac{(b+c-a)\,(a+b-c)}{2(c+a-b)}\right\}^{\frac{1}{2}}. \qquad (16)$$

$$z=\pm\left\{\frac{(c+a-b)\,(b+c-a)}{2(a+b-c)}\right\}^{\frac{1}{2}}. \qquad (17)$$

This example is a good illustration of the beautiful method of deriving one quantity from another, of a similar nature, by simply permuting.

4. Given $\left\{\begin{array}{l} x+\dfrac{1}{y}=a, \\ y+\dfrac{1}{x}=b, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x=\dfrac{a}{2}\pm\sqrt{\dfrac{a^2}{4}-\dfrac{a}{b}}. \\ y=\dfrac{b}{2}\pm\sqrt{\dfrac{b^2}{4}-\dfrac{b}{a}}. \end{array}\right.$

The two equations of this example are also symmetrical, and consequently allow of permutation.

5. Given $\left\{\begin{array}{l} x^2+y^2=a, \\ xy=b, \end{array}\right\}$ to find the values of x and y.

Ans. $\left\{\begin{array}{l} x=\pm\frac{1}{2}(a+2b)^{\frac{1}{2}}\pm\frac{1}{2}(a-2b)^{\frac{1}{2}}. \\ y=\pm\frac{1}{2}(a+2b)^{\frac{1}{2}}\mp\frac{1}{2}(a-2b)^{\frac{1}{2}}. \end{array}\right.$

The equations of this example allow of an interchange of x and y.

6. Given $\left\{\begin{array}{l} x^2+y^2+x+y=a, \\ x^2-y^2+x-y=b, \end{array}\right\}$ to find x and y.

$$Ans. \left\{\begin{array}{l} x=-\frac{1}{2}\pm\frac{1}{2}(1+2a+2b)^{\frac{1}{2}}. \\ y=-\frac{1}{2}\pm\frac{1}{2}(1+2a-2b)^{\frac{1}{2}}. \end{array}\right.$$

7. Given $\left\{\begin{array}{l} x^2+y^2=a, \\ xz=b, \\ yz=c, \end{array}\right\}$ to find x, y, and z.

$$Ans. \left\{\begin{array}{l} x=b\sqrt{\frac{a}{b^2+c^2}}. \\ y=c\sqrt{\frac{a}{b^2+c^2}}. \\ z=\sqrt{\frac{b^2+c^2}{a}}. \end{array}\right.$$

The above are not symmetrical, still they allow of the interchange of x and y, and of b and c. But z and a must remain unchanged.

(121.) QUESTIONS WHICH REQUIRE FOR THEIR SOLUTION A KNOWLEDGE OF QUADRATIC EQUATIONS.

1. A widow possessed 13000 dollars, which she divided into two parts, and placed them at interest, in such a manner that the incomes from them were equal. If she had put out the first portion at the same rate as the second, she would have drawn for this part 360 dollars interest; and if she had placed the second out at the same rate as the first, she would have drawn for it 490 dollars interest. What were the two rates of interest?

Let $x =$ the rate per cent. of the first part.
Let $y =$ the rate per cent. of the second part.

Now, since the incomes of the two parts were equal, they must have been to each other reciprocally as x to y.

We must therefore divide \$13000 into two parts which shall be to each other as y to x; that is, the first part must be $\frac{y}{x+y}$ of the whole, and the second part must be $\frac{x}{x+y}$ of the whole.

Therefore,
$$\frac{13000y}{x+y}=\text{the first part.}$$
$$\frac{13000x}{x+y}=\text{the second part.}$$

The interest on these parts, at y and x per cent., respectively, is

$$\frac{130y^2}{x+y} \text{ and } \frac{130x^2}{x+y}$$

Hence, by the conditions of the question, we have

$$\frac{130y^2}{x+y}=360. \qquad (1)$$

$$\frac{130x^2}{x+y}=490. \qquad (2)$$

Dividing (2) by (1), we get

$$\frac{x^2}{y^2}=\frac{49}{36}. \qquad (3)$$

Extracting the square root of (3), we have

$$\frac{x}{y}=\frac{7}{6}. \qquad (4)$$

Subtracting (1) from (2), we have

$$\frac{130(x^2-y^2)}{x+y}=130. \qquad (5)$$

Dividing both numerator and denominator of the left-hand member of (5) by $x+y$, and also dividing both members by 130, we get

$$x-y=1. \qquad (6)$$

Dividing (6) by y, we find

$$\frac{x}{y}-1=\frac{1}{y}. \qquad (7)$$

Subtracting (7) from (4), we have

$$1=\frac{7}{y}-\frac{1}{y}. \qquad (8)$$

Clearing (8) of fractions, we obtain

$$6y=7y-6. \qquad (9)$$

Therefore, $y=6$. (10)

Adding (10) and (6), we get

$$x=7.$$

Therefore, the rate per cent. of the first part was 7, and that of the second part was 6.

2. A certain capital is on interest at 4 per cent.; if we multiply the number of dollars in the capital, by the number of dollars in the interest for 5 months, we obtain \$117041⅔. What is the capital? *Ans.* \$2650.

3. There are two numbers, one of which is greater than the other by 8, and whose product is 240. What numbers are they? *Ans.* 12 and 20.

4. The sum of two numbers is $=a$, their product $=b$. What numbers are they? *Ans.* $\frac{a+\sqrt{(a^2-4b)}}{2}, \frac{a-\sqrt{(a^2-4b)}}{2}$

5. It is required to find a number, such, that if we multiply its third part by its fourth, and to the product add five times the number required, the sum exceeds the number 200 by as much as the number sought is less than 280. *Ans.* 48.

6. A person being asked his age, answered, "My mother was 20 years old when I was born, and her age, multiplied by mine, exceeds our united ages by 2500." What was his age?

Ans. 42 years.

7. Determine the fortunes of three persons, A., B., C., from the following data: For every \$5 which A. possesses B. has \$9, and C. \$10. Farther, if we multiply A.'s money (expressed in dollars, and considered merely as a number) by B.'s; and B.'s money by C.'s, and add both products to the united fortunes of all three, we shall have 8832. How much had each?

Ans. A. \$40, B. \$72, C. \$80.

8. A person buys some pieces of cloth, at equal prices, for \$60. Had he received three more pieces for the same sum, each piece would have cost him \$1 less. How many pieces did he buy?

Ans. 12 pieces.

9. Two travellers, A. and B., set out at the same time, from two different places, C. and D.; A. from C. to D.; and B. from D. to C. On the way they met, and it then appears that A. had already gone 30 miles more than B., and, according to the rate at which they travel, A. calculates that he can reach the place D. in 4 days, and that B. can arrive at the place C. in 9 days. What is the distance between C. and D.? *Ans.* 150 miles.

10. Divide the number 60 into two such parts, that their product may be to the sum of their squares, in the ratio of 2 to 5.

Ans. 20 and 40.

11. A grazier bought as many sheep as cost him \$150, and after reserving 15 out of the number, he sold the remainder for \$135, and gained \$$\frac{1}{2}$ a head. How many sheep did he buy?

Ans. 75 sheep.

12. What number is that, which, when divided by the product of its two digits, the quotient is 3; and if 18 be added to it, the digits will be inverted? *Ans.* 24.

13. Two partners, A. and B., gained \$140 by trade; A.'s money was 3 months in trade, and his gain was \$60 less than his stock; and B.'s money, which was \$50 more than A.'s, was in trade 5 months. What was A.'s stock? *Ans.* \$100.

14. It is required to divide the number 24 into two such parts, that their product may be equal to 35 times their difference. What are the parts? *Ans.* 10 and 14.

15. A company at a tavern had \$60 to pay for their reckoning; but, before the bill was settled, two of them left the room, and then those who remained had \$1 apiece more to pay than before. How many were there in the company? *Ans.* 12.

16. There are two numbers whose difference is 15, and half their product is equal to the cube of the less number. What are those numbers? *Ans.* 3 and 18.

17. A merchant bought a certain number of pieces of cloth for \$200, which he sold again at \$10½ per piece, and gained by the bargain as much as one piece cost him. What was the number of pieces? *Ans.* 20 pieces.

18. Divide 16 into two such parts, so that the product of the two parts added to the sum of their squares may equal 208.

Ans. The parts are 4 and 12.

19. Find two numbers which shall be to each other as m to n, and the sum of their squares equal to a.

$$Ans.\ m \times \left(\frac{a}{m^2+n^2}\right)^{\frac{1}{2}};\ n \times \left(\frac{a}{m^2+n^2}\right)^{\frac{1}{2}}.$$

20. Find two numbers which shall be to each other as m to n, and the difference of their squares equal to a.

$$Ans.\ m \times \left(\frac{a}{m^2-n^2}\right)^{\frac{1}{2}} \cdot\ n \times \left(\frac{a}{m^2-n^2}\right)^{\frac{1}{2}}.$$

21. The sum of the squares of two numbers is equal to a, and the difference of their squares is equal to b. Find these numbers.

Ans. $\frac{1}{2}(2a+2b)^{\frac{1}{2}}$; $\frac{1}{2}(2a-2b)^{\frac{1}{2}}$.

22. Find a number, such, that its square diminished by $\frac{1}{4}$ may equal $\frac{3}{4}$ of itself. *Ans.* 1, or $-\frac{1}{4}$.

23. What numbers are they, whose sum, product, and difference of their squares are all equal to one another?

Ans. $\frac{1}{2}(3\pm\sqrt{5})$; $\frac{1}{2}(1\pm\sqrt{5})$.

24. A. and B. together, agree to dig 100 rods of ditch for $100. That part of the ditch on which A. was employed was more difficult of excavation than the part on which B. was employed. It was therefore agreed that A. should receive for each rod 25 cents more than B. received for each rod which he dug. How many rods must each dig, and at what prices, so that each may receive just $50?

Let x=number of dollars received by B. for each rod which he dug. Then will $x+\frac{1}{4}$=the number of dollars received by A. for each rod which he dug. Since 50 is the number of dollars which each received, we have

$$\frac{50}{x+\frac{1}{4}}=\frac{200}{4x+1}=\text{number of rods dug by A.,}$$

$$\frac{50}{x}=\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{B.}$$

These added must give the whole number of rods. Hence we have this condition,

$$\frac{200}{4x+1}+\frac{50}{x}=100.$$

Clearing of fractions and transposing, &c., we find

$$8x^2-6x=1\text{; or, }x^2-\tfrac{3}{4}x=\tfrac{1}{8}.$$

Hence, $$x=\tfrac{3}{8}\pm\tfrac{1}{8}\sqrt{17}.$$

Since $\frac{1}{8}\sqrt{17}$ is greater than $\frac{3}{8}$, we must reject the negative value of this surd, otherwise we should have a negative value of x; that is, B. would receive a negative price for his work, which would be absurd.

Using only the plus sign of the surd, we have

$$x=\tfrac{3}{8}+\tfrac{1}{8}\sqrt{17}=\tfrac{1}{8}(3+\sqrt{17})$$

for the number of dollars which B. received per rod;

$$x+\tfrac{1}{4}=\tfrac{1}{4}+\tfrac{3}{8}+\tfrac{1}{8}\sqrt{17}=\tfrac{1}{8}(5+\sqrt{17})$$

for the number of dollars which A. received per rod.

We also find

$$50\div\tfrac{1}{8}(5+\sqrt{17})=\frac{400}{5+\sqrt{17}}=50(5-\sqrt{17})$$

for the number of rods dug by A;

and $$50\div\tfrac{1}{8}(3+\sqrt{17})=\frac{400}{3+\sqrt{17}}=50(\sqrt{17}-3)$$

for the number of rods dug by B.

PROPERTIES OF THE ROOTS OF QUADRATIC EQUATIONS.

(122.) We have seen that all quadratic equations can be reduced to this general form,

$$x^2+ax=b. \qquad (1)$$

This, when solved by the Rule under Art. 118, gives

$$x=-\frac{a}{2}\pm\sqrt{\frac{a^2}{4}+b}. \qquad (2)$$

Therefore the two values of x are

$$-\frac{a}{2}+\sqrt{\frac{a^2}{4}+b}, \qquad (3)$$

$$-\frac{a}{2}-\sqrt{\frac{a^2}{4}+b}. \qquad (4)$$

(123.) Now, since $\frac{a^2}{4}=\left(\frac{a}{2}\right)^2$ is always positive for all real values of a, it follows that the sign of the expression $\frac{a^2}{4}+b$, depends upon the value of b.

(124.) When b is positive, or when b is negative and less than $\frac{a^2}{4}$, then will $\frac{a^2}{4}+b$ be positive, and consequently $\sqrt{\frac{a^2}{4}+b}$ will be real.

(125.) When b is negative and numerically greater than $\frac{a^2}{4}$, then $\frac{a^2}{4}+b$ will be negative, and consequently $\sqrt{\frac{a^2}{4}+b}$ will be *imaginary.*

CASE I.

When $\sqrt{\frac{a^2}{4}+b}$ *is real.*

1. If a is positive, and $\frac{a}{2}$ is numerically greater than $\sqrt{\frac{a^2}{4}+b}$, then will both values of x be *real* and *negative.*

2. When a is either positive or negative, and $\frac{a}{2}$ is numerically less than $\sqrt{\frac{a^2}{4}+b}$, then will both values of x be *real,* the one *positive,* and the other *negative.*

3. When a is negative and $\frac{a}{2}$ is numerically greater than $\sqrt{\frac{a^2}{4}+b}$, then both values of x will be *real* and *positive.*

CASE II.

When $\sqrt{\frac{a^2}{4}+b}$ *is imaginary.*

In this case both values of x are *imaginary* for all values of a.

(126.) When b is negative and numerically equal to $\frac{a^2}{4}$, then both values of x become $=-\frac{a}{2}$.

(127.) If we add together the two values of x, we have

$$\left(-\frac{a}{2}+\sqrt{\frac{a^2}{4}+b}\right)+\left(-\frac{a}{2}-\sqrt{\frac{a^2}{4}+b}\right)=-a.$$

If we multiply them, we find

$$\left(-\frac{a}{2}+\sqrt{\frac{a^2}{4}+b}\right)\times\left(-\frac{a}{2}-\sqrt{\frac{a^2}{4}+b}\right)=-b.$$

From which we see,

That the sum of the roots of the quadratic equation $x^2+ax=b$ *is equal to* $-a$.

And the product of the roots is equal to $-b$.

(128.) We have seen that every quadratic equation, when solved, gives two values for the unknown quantity. These values will both satisfy the algebraic conditions, and sometimes they will both satisfy the particular conditions of the problem, but in most cases but one value of the unknown is applicable to the problem; and the value to be used must be determined from the nature of the question.

We will illustrate this principle by the solution of some particular questions.

1. Find a number, such, that its square being subtracted from five times the number, shall give 6 for remainder.

Let x=the number sought.

Then, by the conditions of the question, we have

$$5x-x^2=6. \qquad (1)$$

Changing all the signs of (1), it becomes

$$x^2-5x=-6, \qquad (2)$$

which, when solved by the rule for quadratics, gives

$$x^2=\frac{5\pm1}{2}=3, \text{ or } 2.$$

Taking the first value, $x=3$, we find its square to be 9.
Five times this value of x is $5\times3=15$.
And $15-9=6$; therefore the number 3 satisfies the question.

The number 2 will satisfy it equally well, since its square $=4$, which subtracted from five times $2=10$, gives for the remainder 6.

2. Find a number, such, that when added to 6, and the sum multiplied by the number, the product will equal the number diminished by 6.

Let $x=$ the number sought; then by the conditions of the question we have

$$(x+6)x=x-6. \qquad (1)$$

Expanding and collecting terms, we find

$$x^2+5x=-6. \qquad (2)$$

This solved, gives $x=\dfrac{-5\pm1}{2}=-3, \text{ or } -2.$

Here, as in the last question, we find that both values of x will satisfy our question.

If we take the first value, $x=-3$, we find that the number -3 added to 6 gives 3, which multiplied by -3 gives -9; and this is the same as -3 diminished by 6.

If we take the second value, $x=-2$, we find that the number -2 added to 6 gives 4, which multiplied by -2 gives -8; and this is the same as -2 diminished by 6.

3. Find a number which subtracted from its square shall give 6 for remainder.

Let x=the number; then we have

$$x^2-x=6.$$

This gives $$x=\frac{1\pm5}{2}=3, \text{ or } -2.$$

If we take 3 for the number, its square is 9, from which subtracting 3, we have 6.

Again, taking -2 for the number, its square is 4, from which subtracting -2, we have 6.

So that both values of x satisfy the conditions of the question.

(129.) We will now add a couple of examples for the purpose of illustrating the case in which the roots are imaginary.

1. Find two numbers whose sum is 8, and whose product is 17.

Let x=the less number, then will $8-x$=the greater number.

The product is $(8-x)x=8x-x^2$, which by the conditions of the question is 17.

Therefore we have this equation of condition,

$$x^2-8x=-17.$$

This, solved by the usual rules for quadratics, gives

$$x=4\pm\sqrt{-1}, \text{ for the less number,}$$

and $8-(4\pm\sqrt{-1})=4\mp\sqrt{-1}$, for the greater number.

Therefore, the numbers are $\begin{cases} 4\pm\sqrt{-1}, \\ 4\mp\sqrt{-1}, \end{cases}$

both of which are imaginary; we are therefore authorized to conclude that it is *impossible* to find two numbers whose sum is 8, and product 17.

We may also satisfy ourselves of this as follows: Since the sum of two numbers is 8, they must average just 4; hence the greater must exceed 4 just as much as the less falls short of 4.

Therefore, any two numbers whose sum is 8, may be represented by

$$4+x,$$
$$4-x.$$

Taking their product, we have

$$(4+x)(4-x)=16-x^2.$$

Now, since x^2 is positive for all *real* values of x, it follows that the product $16-x^2$ is always less than 16; that is, *no two real numbers whose sum is* 8 *can be found, such that their product can equal* 17.

If we put the expression for the product, which we have just found equal to 17, we shall have

$$16-x^2=17.$$

Consequently, $x=\pm\sqrt{-1}.$

And, $\left.\begin{array}{l} 4+x=4\pm\sqrt{-1} \\ 4-x=4\mp\sqrt{-1} \end{array}\right\}$ the same values as found by the first method.

These values, although they are imaginary, will satisfy the algebraic conditions of the question; that is, their sum is

$$(4\pm\sqrt{-1})+(4\mp\sqrt{-1})=8,$$

and their product is

$$(4\pm\sqrt{-1})\times(4\mp\sqrt{-1})=17.$$

2. Find two numbers whose sum is 2, and sum of their reciprocals 1.

Denoting the numbers by x and y, we have the following relations:

$$\left.\begin{array}{l} x+y=2, \\ \dfrac{1}{x}+\dfrac{1}{y}=1. \end{array}\right\} \qquad (1)$$

These, solved by the ordinary rules, give

$$\left.\begin{array}{l} x=1\pm\sqrt{-1}, \\ y=1\mp\sqrt{-1}. \end{array}\right\} \qquad (2)$$

Both these values are imaginary, consequently the conditions of the question are absurd.

We may also show the impossibility of this question as follows: The sum being 2, the numbers may be denoted by $1+x$ and $1-x$.

Taking the sum of their reciprocals, we have

$$\frac{1}{1+x}+\frac{1}{1-x},$$

which, when reduced to a common denominator, becomes

$$\frac{2}{1-x^2}.$$

The denominator of this expression being always less than 1, for all real values of x, the expression must exceed 2. Therefore, *it is impossible to find two numbers whose sums shall equal* 2, *and the sum of their reciprocals equal* 1.

(130.) From what has been said, we conclude that, when in the course of the solution of an algebraic problem, we fall upon imaginary quantities, there must be conditions in the problem which are incompatible.

CHAPTER VI.

RATIO AND PROGRESSION.

(131.) By *Ratio* of two quantities, we mean their relation. When we compare quantities by seeing how much greater one is than another, we obtain *arithmetical ratio.* Thus, the arithmetical ratio of 6 to 4 is 2, since 6 exceeds 4 by 2; in the same way the arithmetical ratio of 11 to 7 is 4.

In the relation $$a-c=r, \qquad (1)$$
r is the *arithmetical ratio* of a to c.

The first of the two terms which are compared is called the *antecedent;* the second is called the *consequent.* Thus referring to (1), we have

$a =$ *antecedent,*
$c =$ *consequent,*
$r =$ *ratio.*

From (1), we get by transposition,

$$a=c+r, \qquad (2)$$
$$c=a-r. \qquad (3)$$

Equation (2) shows, *that in an arithmetical ratio the antecedent is equal to the consequent increased by the ratio.*

Equation (3) in like manner shows, *that the consequent is equal to the antecedent diminished by the ratio.*

(132.) When the arithmetical ratio of any two terms is the same as the ratio of any other two terms, the four terms together form an *arithmetical proportion.*

Thus if $a-c=r$; and $a'-c'=r$, then will

$$a-c=a'-c', \tag{4}$$

which relation is an arithmetical proportion, and is read thus: *a is as much greater than c, as a' is greater than c'.*

Of the four quantities constituting an arithmetical proportion, the first and fourth are called the *extremes*, the second and third are called the *means*.

The first and second, together, constitute the *first couplet;* the third and fourth constitute the *second couplet.*

From equation (4), we get by transposing

$$a+c'=a'+c, \tag{5}$$

Which shows, that the sum of the extremes, of an arithmetical proportion, is equal to the sum of the means.

If $c=a'$, then (4) becomes

$$a-a'=a'-c', \tag{6}$$

which changes (5) into

$$a+c'=2a'. \tag{7}$$

So that if three terms constitute an arithmetical proportion, the sum of the extremes will equal twice the mean.

(133.) A series of quantities which increase or decrease by a constant difference, form an *arithmetical progression.* When the series is increasing, it is called an *ascending progression;* when decreasing, it is called a *descending progression.*

Thus, of the two series,

$$1, 3, 5, 7, 9, 11, \&c. \tag{8}$$
$$27, 23, 19, 15, 11, 7, \&c. \tag{9}$$

the first is an ascending progression, whose ratio or *common difference* is 2; the second is a descending progression, whose *common difference* is 4.

(134.) If $a =$ the first term of an ascending arithmetical progression, whose common difference $=d$, the successive terms will be

$$\left.\begin{aligned} a&=\text{first term,}\\ a+d&=\text{second term,}\\ a+2d&=\text{third term,}\\ a+3d&=\text{fourth term,}\\ &\cdots\cdots\cdots\cdots\cdots\\ &\cdots\cdots\cdots\cdots\cdots\\ a+(n-1)d&=n\text{th term.}\end{aligned}\right\} \qquad (10)$$

If we denote the last or nth term by l, we shall have

$$l=a+(n-1)d. \qquad (11)$$

From (11) we readily deduce

$$a=l-(n-1)d, \qquad (12)$$

$$d=\frac{l-a}{n-1}, \qquad (13)$$

$$n=\frac{l-a}{d}+1. \qquad (14)$$

When the progression is descending, we must write $-d$ for d in the above formulas.

Since each term of an arithmetical progression is greater by d than its preceding term, it follows that the term next to the last is as much smaller than the last, as the second term is greater than the first term. Hence, the sum of the second term and the term next to the last is the same as the sum of the first and the last term; and in general, The sum of any two terms equidistant from the extremes is equal to the sum of the extremes, so that the terms will average half the sum of the extremes; consequently, *the sum of all the terms equals half the sum of the extremes multiplied by the number of terms.*

Representing the sum of n terms by s, we have

$$s=\frac{a+l}{2}\times n. \qquad (15)$$

From (15), we easily obtain

$$a=\frac{2s}{n}-l. \qquad (16)$$

$$l=\frac{2s}{n}-a. \qquad (17)$$

$$n=\frac{2s}{a+l}. \qquad (18)$$

Any three of the quantities,

a=first term,
d=common difference,
n=number of terms,
l =last term,
s= sum of all the terms,

being given, the remaining two can be found, which must give rise to 20 different formulas, as given in the following table for Arithmetical Progression.

We have not deemed it necessary to exhibit the particular process of finding each distinct formula of the following table, since they were all derived from the two fundamental ones (11) and (15), by the usual operations of equations not exceeding the second degree.

No.	Given.	Required.	Formulas.	Corr.
1	a, d, n	l	$l=a+(n-1)d$	17
2	a, d, s		$l=-\frac{1}{2}d+\sqrt{2ds+(a-\frac{1}{2}d)^2}$	19
3	a, n, s		$l=\frac{2s}{n}-a$	20
4	d, n, s		$l=\frac{s}{n}+\frac{(n-1)d}{2}$	18
5	a, d, n	s	$s=\frac{1}{2}n[2a+(n-1)d]$	8
6	a, d, l		$s=\frac{l+a}{2}+\frac{(l+a)(l-a)}{2d}$	
7	a, n, l		$s=\frac{1}{2}n(a+l)$	
8	d, n, l		$s=\frac{1}{2}n[2l-(n-1)d]$	5
9	a, n, l	d	$d=\frac{l-a}{n-1}$	
10	a, n, s		$d=\frac{2s-2an}{n(n-1)}$	12
11	a, l, s		$d=\frac{(l+a)(l-a)}{2s-l-a}$	
12	n, l, s		$d=\frac{2nl-2s}{n(n-1)}$	10
13	a, d, l	n	$n=\frac{l-a}{d}+1$	
14	a, d, s		$n=\frac{d-2a}{2d}+\sqrt{\frac{2s}{d}+\left(\frac{2a-d}{2d}\right)^2}$	16
15	a, l, s		$n=\frac{2s}{l+a}$	
16	d, l, s		$n=\frac{2l+d}{2d}-\sqrt{\left(\frac{2l+d}{2d}\right)^2-\frac{2s}{d}}$	14
17	d, n, l	a	$a=l-(n-1)d$	1
18	d, n, s		$a=\frac{s}{n}-\frac{(n-1)d}{2}$	4
19	d, l, s		$a=\frac{1}{2}d+\sqrt{(l+\frac{1}{2}d)^2-2ds}$	2
20	n, l, s		$a=\frac{2s}{n}-l$	3

(135.) From the nature of an arithmetical progression, we discover that if we subtract the common difference from the last term, we shall obtain the term next to the last; if we subtract from the last term twice the common difference, we obtain the second term from the last. Hence the terms of an arithmetical progression will be reversed if we interchange the values of a and l, and at the same time change the sign of d. Thus, the general form of an arithmetical progression is

$$a,\ a+d,\ a+2d, \ldots\ldots\ldots l-2d,\ l-d,\ l.$$

Changing a to l, l to a, and changing the sign of d, we have

$$l,\ l-d,\ l-2d, \ldots\ldots\ldots a+2d,\ a+d,\ a,$$

which is precisely the same progression as the first, with the terms arranged in a reverse order. The above change has, of course, no effect upon the number of terms, nor upon the sum of all the terms.

Therefore, in any of the formulas of the preceding table we are at liberty to make the above-named changes. As an example, we will take from the table formula 2, which is

$$l=-\tfrac{1}{2}d+\sqrt{2ds+(a-\tfrac{1}{2}d)^2}.$$

Now, changing l to a, a to l, and changing the sign of d, it becomes

$$a=\tfrac{1}{2}d+\sqrt{(l+\tfrac{1}{2}d)^2-2ds},$$

which is formula 19.

In the same way, formulas 14 and 16 may be deduced from each other. Such formulas as may be derived from each other by the above changes we shall call correlative formulas. It is evident that some of the formulas of the table have no correlative. Thus, formulas 13 and 15 are not altered by the above changes. Those formulas which have correlative formulas have them referred to in the table, under column headed *Corr.*

EXAMPLES.

1. The first term of an arithmetical progression is 7, the common difference is $\frac{1}{4}$, and the number of terms is 16. What is the last term?

To solve this we take formula 1 from our table, which is

$$l=a+(n-1)d.$$

Substituting the above given values for a, d, and n, we find

$$l=7+\tfrac{1}{4}(16-1)=10\tfrac{3}{4}.$$

2. The first term of an arithmetical progression is $\frac{3}{4}$, the common difference is $\frac{1}{8}$, and the last term is $3\frac{7}{8}$. What is the number of terms?

In this example we take formula 13.

$$n=\frac{l-a}{d}+1;$$

which in this present case becomes

$$n=\frac{3\frac{7}{8}-\frac{3}{4}}{\frac{1}{8}}+1=26.$$

3. One hundred stones being placed on the ground in a straight line, at the distance of 2 yards from each other, how far will a person travel who shall bring them one by one to a basket, placed at 2 yards from the first stone?

In this example $a=4$; $d=4$; $n=100$, which values being substituted in formula 5, give

$$s=50(8+99\times4)=20200 \text{ yards.}$$

which, divided by 1760, the number of yards in one mile, we get

$$s=11 \text{ miles, } 840 \text{ yards.}$$

4. What is the sum of n terms of the progression

$1, 3, 5, 7, 9, \ldots.$? *Ans.* $s=n^2$.

5. What is the sum of n terms of the progression

1, 2, 3, 4, 5, ... ? *Ans.* $s=\frac{n(n+1)}{2}$.

6. A man buys 10 sheep, giving \$1 for the first, \$3 for the second, \$5 for the third, and so increasing in arithmetical progression. What will the last sheep cost at that rate?

By formula 1. *Ans.* \$19.

7. A person travels 25 days, going 11 miles the first day, and 135 the last day; the miles which he travelled in the successive days form an arithmetical progression. How far did he go in the 25 days? By formula 7. *Ans.* 1825 miles.

8. A note becomes due in annual instalments, which are in arithmetical progression, whose common difference is 3; the first payment is 7 dollars, the last payment is 49 dollars. What is the number of instalments? By formula 13. *Ans.* 15.

9. In a triangular field of corn, the number of hills in the successive rows are in arithmetical progression: in the first row there is but one hill, in the last row there are 81 hills; and the whole number of hills in the field is 1681. How many rows are there? By formula 15. *Ans.* 41.

GEOMETRICAL RATIO.

(136.) When we compare quantities by seeing how many times greater one is than another, we obtain *geometrical ratio.* Thus, the geometrical ratio of 8 to 4 is 2, since 8 is 2 times as great as 4. Again, the geometrical ratio of 15 to 3 is 5.

In the relation,

$$\frac{a}{c}=r, \qquad (1)$$

r is the geometrical ratio of a to c.

As in arithmetical ratio,

a=*antecedent.*
c=*consequent.*
r=*ratio.*

From (1), we get by reduction,

$$a=cr, \tag{2}$$

$$c=\frac{a}{r}. \tag{3}$$

Equation (2) shows, *that in a geometrical ratio the antecedent is equal to the consequent multiplied by the ratio.*

Equation (3) shows, *that the consequent is equal to the antecedent divided by the ratio.*

(137.) When the geometrical ratio of any two terms is the same as the ratio of any other two terms, the four terms together form a *geometrical proportion.*

Thus if $\frac{a}{c}=r$; and $\frac{a'}{c'}=r$, then will

$$\frac{a}{c}=\frac{a'}{c'}, \tag{4}$$

which relation is a geometrical proportion, and is generally written thus:

$$a : c :: a' : c', \tag{5}$$

which is read as follows: *a is to c, as a′ is to c′.*

Of the four quantities which constitute a geometrical proportion, as in arithmetical proportion, the first and fourth are called the *extremes*, the second and third are called the *means.*

The first and second constitute the *first couplet;* the third and fourth constitute the *second couplet.*

From equation (5), or its equivalent (4), we find

$$ac'=a'c. \tag{6}$$

Which shows, that the product of the extremes of a geometrical proportion is equal to the product of the means.

If $c=a'$ then (5) becomes

$$a : a' :: a' : c', \tag{7}$$

which changes (6) into

$$ac' = a'^2, \tag{8}$$

so that, if three terms constitute a geometrical proportion, the product of the extremes will equal the square of the mean.

(138.) Quantities are said to be in proportion *by inversion*, or *inversely*, when the consequents are taken as antecedents, and the antecedents as consequents.

From (5), or its equivalent (4), which is

$$\frac{a}{c} = \frac{a'}{c'}, \tag{9}$$

we have by inverting both terms

$$\frac{c}{a} = \frac{c'}{a'}.$$

Therefore, by Art. 137,

$$c : a :: c' : a'. \tag{10}$$

Which shows, that if four quantities are in proportion, they will be in proportion by inversion.

(139.) Quantities are in proportion by *alternation*, or *alternately*, when the antecedents form one of the couplets, and the consequents form the other.

Resuming (4),

$$\frac{a}{c} = \frac{a'}{c'}. \tag{11}$$

Multiplying both terms of (11) by $\frac{c}{a'}$, it will become

$$\frac{a}{a'} = \frac{c}{c'}.$$

Therefore, by Art. 137,

$$a : a' :: c : c'. \tag{12}$$

Which shows, that if four quantities are in proportion, they will be so by alternation.

(140.) Quantities are in proportion by *composition*, when the sum of the antecedent and consequent is compared either with antecedent or consequent.

Resuming (4),

$$\frac{a}{c}=\frac{a'}{c'}. \qquad (13)$$

If to (13), we add the terms of the following equation, $\frac{c}{c}=\frac{c'}{c'}$, each of whose members is equal to unity, we have

$$\frac{a+c}{c}=\frac{a'+c'}{c'}.$$

Therefore, by Art. 137,

$$a+c : c :: a'+c' : c'. \qquad (14)$$

Which shows, that if four quantities are in proportion, they will be so by composition.

(141.) Quantities are said to be in proportion by *division*, when the difference of the antecedent and consequent is compared with either antecedent or consequent.

If we subtract the equation $\frac{c}{c}=\frac{c'}{c'}$, each member of which is equal to 1, from equation (4), we find

$$\frac{a-c}{c}=\frac{a'-c'}{c'}. \qquad (15)$$

Therefore, by Art. 137, we have

$$a-c : c :: a'-c' : c'. \qquad (15)$$

Which shows, that if four quantities are in proportion, they will be so by division.

Equation (4) is

$$\frac{a}{c}=\frac{a'}{c'}.$$

Raising each member to the nth power, we have

$$\frac{a^n}{c^n}=\frac{a'^n}{c'^n}.$$

Therefore, since the above equation is true for all values of n, by Art. 137, we have

$$a^n : c^n :: a'^n : c'^n. \qquad (16)$$

Which shows, that if four quantities are in proportion, like powers or roots of these quantities will also be in proportion.

If we have

$$\left.\begin{array}{l} a : c :: a' : c', \\ a : c :: a'' : c'', \\ a : c :: a''' : c''', \\ \quad\&c. \qquad \&c. \end{array}\right\} \qquad (17)$$

we have, by alternation, Art. 139,

$$\left.\begin{array}{l} a : a' :: c : c', \\ a : a'' :: c : c'', \\ a : a''' :: c : c''', \\ \quad\&c. \qquad \&c. \end{array}\right\}$$

Therefore, by inversion, Art. 138, we have

$$\left.\begin{array}{l} \frac{a'}{a}=\frac{c'}{c}, \\ \frac{a''}{a}=\frac{c''}{c}, \\ \frac{a'''}{a}=\frac{c'''}{c}, \\ \&c. \quad \&c. \\ \frac{a}{a}=\frac{c}{c}. \end{array}\right\} \qquad (18)$$

We also have $\frac{a}{a}=\frac{c}{c}.$

Taking the sum of equations (18), we have

$$\frac{a+a'+a''+a'''+\&c.}{a}=\frac{c+c'+c''+c'''+\&c.}{c} \qquad (19)$$

Therefore, we have

$$a+a'+a''+a'''+\&c. : a :: c+c'+c''+c'''+\&c. : c.$$

Which shows, that if any number of quantities are proportional, the sum of all the antecedents will be to any one antecedent, as the sum of all the consequents is to its corresponding consequent

(142.) If we have

$$a : c :: a' : c',$$
$$a'' : c'' :: a''' : c''',$$

then we find

$$\frac{a}{c}=\frac{a'}{c'}, \qquad (21)$$

$$\frac{a''}{c''}=\frac{a'''}{c'''}. \qquad (22)$$

Multiplying together the equations (21) and (22), we have

$$\frac{a\times a''}{c\times c''}=\frac{a'\times a'''}{c'\times c'''} \qquad (23)$$

Therefore, by Art. 137, we have

$$a\times a'' : c\times c'' :: a'\times a''' : c'\times c''', \qquad (24)$$

Which shows, that if there be two sets of proportional quantities, the products of the corresponding terms will be proportional.

(143.) A series of quantities which increase or decrease by a constant multiplier, form a *geometrical progression.* When the series is increasing, that is, when the constant multiplier exceeds a unit, it is called an *ascending progression;* when decreasing, or when the constant multiplier is less than a unit, then it is called a *descending progression.*

Thus, of the two series,

1, 3, 9, 27, 81, 243, &c., (25)
256, 128, 64, 32, 16, 8, &c., (26)

the first is an ascending progression, whose constant multiplier or *ratio* is 3; the second is a descending progression, whose ratio is $\frac{1}{2}$.

(144.) If a is the first term of a geometrical progression, whose ratio $=r$, the successive terms will be

$$\left.\begin{array}{r}a=\text{first term,}\\ ar=\text{second term,}\\ ar^2=\text{third term,}\\ ar^3=\text{fourth term,}\\ \ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots\\ ar^{n-1}=n\text{th term.}\end{array}\right\}\qquad(27)$$

If we denote the last or nth term by l, we shall have

$$l=ar^{n-1}.\qquad(28)$$

If we represent the sum of n terms of a geometrical progression by s, we shall have

$$s=a+ar+ar^2+ar^3+\ldots\ldots+ar^{n-2}+ar^{n-1}.\qquad(29)$$

Multiplying all the terms of (29) by the ratio r, we have

$$rs=ar+ar^2+ar^3+ar^4+\ldots\ldots+ar^{n-1}+ar^n.\qquad(30)$$

Subtracting (29) from (30), we get

$$(r-1)s=a(r^n-1).\qquad(31)$$

Therefore,

$$s=a\left\{\frac{r^n-1}{r-1}\right\},\qquad(32)$$

Any three of the quantities

$a=$first term,
$r=$ratio,
$n=$number of terms,
$l=$last term,
$s=$sum of all the terms,

being given, the remaining two can be found, which, as in arithmetical progression, must give rise to 20 different formulas, as given in the following table for GEOMETRICAL PROGRESSION.

No.	Given.	Required.	Formulas.	Corr.
1	a, r, n	l	$l=ar^{n-1}$	9
2	a, r, s		$l=\frac{a+(r-1)s}{r}$	11
3	a, n, s		$l(s-l)^{n-1}-a(s-a)^{n-1}=0$	12
4	r, n, s		$l=\frac{(r-1)sr^{n-1}}{r^n-1}$	10
5	a, r, n	s	$s=\frac{a(r^n-1)}{r-1}$	8
6	a, r, l		$s=\frac{rl-a}{r-1}$	
7	a, n, l		$s=\frac{l^{\frac{n}{n-1}}-a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}}-a^{\frac{1}{n-1}}}$	
8	r, n, l		$s=\frac{l(r^n-1)}{(r-1)r^{n-1}}$	5
9	r, n, l	a	$a=\frac{l}{r^{n-1}}$	1
10	r, n, s		$a=\frac{(r-1)s}{r^n-1}$	4
11	r, l, s		$a=rl-(r-1)s$	2
12	n, l, s		$a(s-a)^{n-1}-l(s-l)^{n-1}=0$	3
13	a, n, l	r	$r=\left(\frac{l}{a}\right)^{\frac{1}{n-1}}$	
14	a, n, s		$r^n-\frac{s}{a}r+\frac{s-a}{a}=0$	16
15	a, l, s		$r=\frac{s-a}{s-l}$	
16	n, l, s		$r^n-\frac{s}{s-l}r^{n-1}+\frac{l}{s-l}=0$	14
17	a, r, l	n	$n=\frac{\log l-\log a}{\log r}+1$	
18	a, r, s		$n=\frac{\log[a+(r-1)s]-\log a}{\log r}$	20
19	a, l, s		$n=\frac{\log l-\log a}{\log(s-a)-\log(s-l)}+1$	
20	r, l, s		$n=\frac{\log l-\log[rl-(r-1)s]}{\log r}+1$	18

(145.) All the formulas of the above table are easily drawn from the conditions of (28) and (32), which conditions correspond with formulas (1) and (5), except the last four, which involve logarithms.

If in a geometrical progression we change a to l, l to a, and r to $r^{-1}=\frac{1}{r}$, the progression will remain the same as before, taken in a reverse order. These changes being made in the formulas of the preceding table, we shall discover that some of the formulas, as in arithmetical progression, have correlative formulas. Those having correlative formulas have them referred to in the table, under column headed *Corr.*

EXAMPLES.

1. The first term of a geometrical progression is 5, the ratio 4, the number of terms is 9. What is the last term?

Formula (1), which is $l=ar^n$, gives

$$l=5\times 4^8=327680.$$

2. The first term of a geometrical progression is 4, the ratio is 3, the number of terms is 10. What is the sum of all the terms?

Formula (5), which is $s=\frac{a(r^n-1)}{r-1}$, gives

$$s=\frac{4(3^{10}-1)}{2}=118096.$$

3. The last term of a geometrical progression is $106\frac{403}{512}$, the ratio is $\frac{3}{2}$, the number of terms 8. What is the first term?

Formula (9), which is $a=\frac{l}{r^{n-1}}$, gives

$$a=\frac{106\frac{403}{512}}{(\frac{3}{2})^7}=6\frac{1}{4}.$$

(146.) When the progression is descending, the ratio is less than one, and if we suppose the series extended to an infinite number of terms, the last term may be taken $l=0$, which causes formula (6) to become

$$s=\frac{a}{1-r}, \qquad (33)$$

Which shows that the sum of an infinite number of terms of a descending geometrical progression is equal to its first term, divided by one diminished by the ratio.

EXAMPLES.

1. What is the sum of the infinite progression

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\text{\&c.?}$$

In this example $a=1$, $r=\frac{1}{2}$, and (33) becomes

$$s=\frac{1}{1-\frac{1}{2}}=2.$$

2. What is the value of 0.33333 &c., or which is the same thing, of the infinite series $\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+$&c.?

Here $a=\frac{3}{10}$, $r=\frac{1}{10}$, and (33) gives

$$s=\frac{\frac{3}{10}}{1-\frac{1}{10}}=\tfrac{3}{9}=\tfrac{1}{3}.$$

3. What is the value of 0.12121212 &c., or which is the same, of $\frac{12}{100}+\frac{12}{10000}+\frac{12}{1000000}+$&c.?

In this example $a=\frac{12}{100}$, $r=\frac{1}{100}$, and (33) gives

$$s=\frac{\frac{12}{100}}{1-\frac{1}{100}}=\tfrac{12}{99}=\tfrac{4}{33}.$$

4. What is the sum of the infinite series

$1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+$&c.? *Ans.* $\frac{3}{2}$.

5. What is the sum of the infinite series

$\frac{1}{5}+\frac{1}{25}+\frac{1}{125}$ &c.? *Ans.* $\frac{1}{4}$.

6. A person sowed a peck of wheat, and used the whole crop for seed the following year; the produce of the 2d year again for seed the 3d year, and so on. If, in the last year his crop is 1048576 pecks, how many pecks did he raise in all, allowing the increase to have been in a four-fold ratio?

By formula 6. *Ans.* 1398101 pecks.

7. A king in India, named Sheran, wished, according to the Arabic author Asephad, that Sessa, the inventor of chess, should himself choose a reward. He requested the grains of wheat which arise when 1 is calculated for the first square of the board, 2 for the second square, 4 for the third, and so on; reckoning for each of the 64 squares of the board, twice as many grains as for the preceding. When it was calculated, to the astonishment of the king, it was found to be an enormous number. What was it? By formula 5.

Ans. 18446744073709551615 grains.

HARMONICAL PROPORTION.

(147.) Three quantities are in harmonical proportion, when the first has the same ratio to the third, as the difference between the first and second has to the difference between the second and third.

Four quantities are in harmonical proportion, when the first has the same ratio to the fourth, as the difference between the first and second has to the difference between the third and fourth. Thus, if

$$a : c :: a-b : b-c, \qquad (2)$$

then will the three quantities a, b, c, be in harmonical proportion. If

$$a : d :: a-b : c-d, \qquad (2)$$

then also will the four quantities a, b, c, and d be in harmonical proportion.

Multiplying means and extremes of (1), we have

$$ab-ac=ac-bc, \tag{3}$$

which by transposition becomes

$$ab+bc=2ac. \tag{4}$$

In a similar way equation (2) gives

$$ac+bd=2ad. \tag{5}$$

Suppose a, b, c, d, e, &c., to be in harmonical progression; then from (4) we have

$$\left.\begin{array}{c} bc+ab=2ac \\ cd+bc=2bd \\ de+cd=2ce \\ \text{\&c.} \end{array}\right\} \tag{6}$$

Dividing the first of (6) by abc, the second by bcd, and the third by cde, &c., we find

$$\left.\begin{array}{c} \frac{1}{a}+\frac{1}{c}=\frac{2}{b} \\ \frac{1}{b}+\frac{1}{d}=\frac{2}{c} \\ \frac{1}{c}+\frac{1}{e}=\frac{2}{d} \\ \text{\&c.} \end{array}\right\} \tag{7}$$

From which we see that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}$, &c., are in arithmetical progression.

Hence, the reciprocals of any number of terms in harmonical progression are in arithmetical progression; and conversely, the reciprocals of the terms of any arithmetical progression must be in harmonical progression.

The reciprocals of the arithmetical series 1, 2, 3, 4, 5, 6, are $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, whose numerators, when reduced to a common denominator, are 60, 30, 20, 15, 12, 10, which by the above property must be in harmonical progression.

If six musical strings, of equal tension and thickness, have their lengths in proportion to the above numbers, they will, when sounded together, produce more perfect harmony than could be produced by strings of different lengths; and hence we see the propriety of calling this kind of relation *harmonical* or *musical* proportion.

If we take the arithmetical mean, the geometrical mean, and the harmonical mean, of any two numbers, these three means will be in geometrical proportion.

Let a and b be any two numbers, then will

$$\tfrac{1}{2}(a+b) = \text{their arithmetical mean},$$
$$\sqrt{ab} = \text{“ geometrical ”}$$
$$\frac{2ab}{a+b} = \text{“ harmonical ”}$$

And we evidently have

$$\tfrac{1}{2}(a+b) : \sqrt{ab} :: \sqrt{ab} : \frac{2ab}{a+b}.$$

That is,

The geometrical mean, between the arithmetical mean and the harmonical mean of two quantities, is the same as the geometrical mean of the quantities themselves.

CHAPTER VII.

PROBLEMS GENERALIZED.

(148.) Under Articles 67 and 74 we have given a few cases of the method of generalizing the solution of problems. We propose now to continue this subject.

Thus far, in most cases, we have made use of numerical values to represent the known quantities. Now it is obvious that if we use letters to represent these known quantities, our operations will be general, and of course applicable to all questions involving similar conditions.

Under this point of view, we see that Algebra furnishes us with concise and easy methods for determining general propositions and rules of operation.

To illustrate this, we will deduce some propositions involving two unknown quantities, x and y.

Putting $s=$their sum, and $d=$their difference, we have

$$x+y=s. \tag{1}$$

$$x-y=d. \tag{2}$$

Adding (2) to (1), we get

$$2x=s+d, \tag{3}$$

Subtracting (2) from (1), we get

$$2y=s-d. \tag{4}$$

Dividing each member of (3) and (4) by 2, we obtain

$$x=\frac{s+d}{2}, \text{ or } x=\frac{s}{2}+\frac{d}{2}=\text{greater number.}$$

$$y=\frac{s-d}{2}, \text{ or } y=\frac{s}{2}-\frac{d}{2}=\text{less number.}$$

Translating these results into common language, we have the following

PROPOSITION.

Half the difference of two quantities, added to half their sum, is equal to the greater.

Half the difference of two quantities, subtracted from half their sum, is equal to the less.

Again, if we take the squares of (1) and (2), we shall have

$$x^2+2xy+y^2=s^2, \tag{5}$$
$$x^2-2xy+y^2=d^2. \tag{6}$$

These give the following

PROPOSITIONS.

I. The square of the sum of two quantities is equivalent to the square of the greater, plus twice the product of both, plus the square of the less.

II. The square of the difference of two quantities is equivalent to the square of the greater, minus twice the product of both, plus the square of the less.

These two propositions have already been noticed under Art. 27.

Subtracting (6) from (5), we find

$$4xy=s^2-d^2.$$

This result translated into common language gives this

PROPOSITION.

Four times the product of any two numbers, is equal to the square of their sum, diminished by the square of their difference.

Again, equations (5) and (6) by transposition give

$$x^2+y^2=s^2-2xy, \tag{7}$$
$$x^2+y^2=d^2+2xy. \tag{8}$$

Equations (7) and (8) are equivalent respectively to these two

PROPOSITIONS.

I. The sum of the squares of two quantities, is equal to the square of their sum, diminished by twice their product.

II. The sum of the squares of two quantities, is equal to the square of their difference, increased by twice their product.

Example 4, Art. 44; examples 6 and 11, Art. 46; example 7, Art. 47; and example 4, Art. 48, give the following expressions:

$$(1)\qquad \frac{m^3-y^3}{m-y}=m^2+y^2+my.$$

$$(2)\qquad \left(\frac{a+b}{2}\right)^2+\left(\frac{a-b}{2}\right)^2=\frac{a^2+b^2}{2}.$$

$$(3)\qquad \frac{1}{x-y}+\frac{1}{x+y}=\frac{2x}{x^2-y^2}.$$

$$(4)\qquad \frac{1}{x-y}-\frac{1}{x+y}=\frac{2y}{x^2-y^2}.$$

$$(5)\qquad \frac{a+b}{2}\times\frac{a-b}{2}=\frac{a^2-b^2}{4}.$$

If we translate these five expressions, we shall obtain the following general

PROPOSITIONS.

I. The difference of the cubes of two quantities, divided by their difference, is equal to the sum of their squares, increased by their product.

II. The square of half the sum of two quantities, increased by the square of half their difference, is equal to half the sum of their squares.

III. The reciprocal of the difference of two quantities, increased by the reciprocal of their sum, is equal to twice the greater quantity divided by the difference of their squares.

IV. The reciprocal of the difference of two quantities, diminished by the reciprocal of their sum, is equal to twice the smaller quantity divided by the difference of their squares.

V. Half the sum of two quantities, multiplied by half their difference, is equal to one-fourth of the difference of their squares.

(149.) From this we conclude that equations are only concise expressions for general propositions.

We will now generalize a few examples, for the purpose of deducing the Arithmetical Rule for Fellowship.

EXAMPLES.

1. Divide the number a into two such parts, that the first shall be to the second as m to n.

Let $\left\{\begin{matrix} mx \\ nx \end{matrix}\right\}$ represent the two parts, which are obviously in ratio of m to n.

Hence we have this condition,

$$mx+nx=a.$$

Dividing both members by $(m+n)$, we find

$$x=\frac{a}{m+n}.$$

Therefore,
$$mx=\frac{ma}{m+n}=\frac{m}{m+n}a,$$

$$nx=\frac{na}{m+n}=\frac{n}{m+n}a.$$

2. Divide the number a into three parts, which shall be to each other respectively as m, n, and p.

Assume, $\left\{\begin{matrix} mx \\ nx \\ px \end{matrix}\right\}$ for the parts.

Taking the sum, we have

$$mx+nx+px=a,$$

and $$x=\frac{a}{m+n+p}.$$

Consequently,

$$mx=\frac{ma}{m+n+p}=\frac{m}{m+n+p}\times a$$

$$nx=\frac{na}{m+n+p}=\frac{n}{m+n+p}\times a$$

$$px=\frac{pa}{m+n+p}=\frac{p}{m+n+p}\times a.$$

From these examples we readily discover how to proceed for a greater number of parts.

The above leads to the following example:

3. To divide a number into three parts, which shall be to each other in the ratio of three numbers.

RULE.

Form three fractions having for their numerators the respective numbers, and a common denominator equal to their sum. These fractions multiplied by the given number will give the respective parts sought.

As an example, let us solve this question:

4. A., B., and C. enter into partnership. A. furnished \$180, B. \$240, and C. \$480. They gained \$300. What is each one's part of the gain?

$180+240+480=900=$ the common denominator.

$$\left.\begin{array}{l}\frac{180}{900}=\frac{1}{5}\\[4pt] \frac{240}{900}=\frac{4}{15}\\[4pt] \frac{480}{900}=\frac{8}{15}\end{array}\right\}\text{equal to the fractional parts of the gain which each receives.}$$

Hence, A. must have $\frac{1}{5}$ of \$300 = \$60
B. " " $\frac{4}{15}$ of \$300 = \$80
C. " " $\frac{8}{15}$ of \$300 = \$160
\$300 verification.

From this we discover that these general solutions lead immediately to the arithmetical rule for Fellowship.

(150.) We will now proceed to show how easy a thing it is to deduce the usual rules of simple interest by means of algebraic operations.

In our operations we shall use the following notation:

p = the principal.
r = the rate per cent., or the interest of \$1 for one year.
n = the number of years the principal is on interest.
a = the amount.

Then it is obvious that we shall have

$$rp = \text{the interest of \$}p\text{ for one year at }r\text{ per cent.} \qquad (1)$$

$$nrp = \text{the interest of \$}p\text{ for }n\text{ years at }r\text{ per cent.} \qquad (2)$$

Hence we have this condition,

$$p + nrp = a. \qquad (3)$$

Dividing both members of (3) by $1+nr$, we have

$$p = \frac{a}{1+nr}. \qquad (4)$$

Equation (3) also readily gives

$$r = \frac{a-p}{np}, \qquad (5)$$

$$n = \frac{a-p}{rp}. \qquad (6)$$

REMARK.—When the time is not an even number of years, we must reduce the months and days, if any, to the proper fractional part of a year. Thus, 3 years and 3 months is the same as as $3\frac{1}{4}$ years. 3 years 3 months and 3 days, is the same as $(3+\frac{1}{4}+\frac{1}{120})=3\frac{31}{120}$ years. And so on for other fractional parts of a year.

The rate per cent. is always a decimal.

Thus at 3 per cent. r becomes 0.03,
" 4 " " " 0.04,
" 5 " " " 0.05,
" 6 " " " 0.06,
" 7 " " " 0.07.

We will now translate these equations.

EQUATION (1).

The interest of a given sum for one year is equal to the principal multiplied by the rate per cent.

EQUATION (2).

The interest of a given sum for any given time is equal to the interest for one year multiplied by the number of years.

EQUATION (3).

The amount of a given sum for a given time at a given rate per cent. is equal to the interest added to the principal.

EQUATION (4).

The principal is equal to the amount divided by the amount of $1 *for the same time and at the same rate per cent.*

EQUATION (5).

The rate per cent. is equal to the interest of the given principal for the given time and the rate per cent., divided by the interest of the same principal for the same time at one per cent.

EQUATION (6).

The number of years is equal to the interest of the given prin-

cipal for the given time and rate per cent., divided by the interest of the same principal at the same rate per cent. for one year.

(151.) The next two examples give several symmetrical expressions possessing considerable interest.

5. A cistern can be filled by three pipes; the first can fill it in a hours, the second in b hours, the third in c hours. In what time will the cistern be filled when all three pipes are open at once?

Let x=the time sought.

Now since the first can fill it in a hours, it can fill $\frac{1}{a}$ part in one hour.

In the same way we see that the second can fill $\frac{1}{b}$ part in 1 hour.

The third can fill $\frac{1}{c}$ part in one hour.

All together will fill $\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ part in one hour.

But by supposition they can all fill $\frac{1}{x}$ part in one hour.

Hence, we have this condition,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{x}. \qquad (1)$$

That is, *The reciprocal of the joint time is equal to the sum of the reciprocals of the individual times.*

This equation readily gives

$$x=\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}. \qquad (2)$$

Hence the joint time is equal to the reciprocal of the sum of the reciprocals of the individual times.

Condition (2) readily takes this form,

$$x=\frac{abc}{bc+ac+ab}. \qquad (3)$$

Translating this expression, it becomes,

Divide the continued product of the respective times which each alone requires, by the sum of their products taken two at a time, and the quotient will give the joint time.

Remark.—If one of the pipes, the first for example, instead of assisting the others to fill the cistern, was constantly drawing off from it, then all that would be necessary to do, so that formula (2) should be true, would be to change the sign of a. Making this change in (2), we get

$$x=\frac{1}{-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}. \qquad (4)$$

When $\frac{1}{a}$ is less than $\frac{1}{b}+\frac{1}{c}$, the value of x is *positive.*

When $\frac{1}{a}$ is greater than $\frac{1}{b}+\frac{1}{c}$, the value of x is *negative.*

When $\frac{1}{a}=\frac{1}{b}+\frac{1}{c}$, the value of x is *infinite.* That is, in this last case the first pipe discharges just as fast as the two others supply. Consequently, the cistern can never become full.

By using a negative exponent to denote the reciprocal, Art. 34, equation (1) becomes

$$x^{-1}=a^{-1}+b^{-1}+c^{-1}. \qquad (5)$$

Similarly, (4) becomes

$$x=\frac{1}{-a^{-1}+b^{-1}+c^{-1}} \qquad (6)$$

We will take another example of symmetrical equations, the solution of which will be found to bear much likeness to the solution of Prob. 3, Art. 120.

6. Given $\left\{\begin{aligned} \frac{xyz}{x+y}&=a, \\ \frac{xyz}{y+z}&=b, \\ \frac{xyz}{z+x}&=c, \end{aligned}\right\}$ to find x, y, and z.

Taking the reciprocals of each member of these equations, we have

$$\frac{x+y}{xyz}=\frac{1}{a},$$
$$\frac{y+z}{xyz}=\frac{1}{b},$$
$$\frac{z+x}{xyz}=\frac{1}{c}.$$

Dividing the numerator and denominator of the left-hand members of each, we finally obtain

$$\frac{1}{yz}+\frac{1}{zx}=\frac{1}{a}, \qquad (1)$$
$$\frac{1}{zx}+\frac{1}{xy}=\frac{1}{b}, \qquad (2)$$
$$\frac{1}{xy}+\frac{1}{yz}=\frac{1}{c}. \qquad (3)$$

Taking half the sum of these three equations, we have

$$\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=\tfrac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right). \qquad (4)$$

Subtracting from (4) successively (1), (2), and (3), we find

$$\frac{1}{xy}=\tfrac{1}{2}\left(-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right), \qquad (5)$$
$$\frac{1}{yz}=\tfrac{1}{2}\left(-\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right), \qquad (6)$$
$$\frac{1}{zx}=\tfrac{1}{2}\left(-\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right). \qquad (7)$$

Taking the continued product of (5), (6), and (7), we have

$$\frac{1}{x^2y^2z^2}=\tfrac{1}{8}\left(-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(-\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)\left(-\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right). \quad (8)$$

Dividing (8) by the square of (6), we have

$$\frac{1}{x^2}=\frac{\tfrac{1}{2}\left(-\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)\left(-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}{-\frac{1}{b}+\frac{1}{c}+\frac{1}{a}}. \quad (9)$$

Reciprocating (9), and extracting the square root, we find

$$x=\pm\left\{\frac{-\frac{1}{b}+\frac{1}{c}+\frac{1}{a}}{\tfrac{1}{2}\left(-\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)\left(-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\right\}^{\frac{1}{2}}. \quad (10)$$

By permuting, we find from (10),

$$y=\pm\left\{\frac{-\frac{1}{c}+\frac{1}{a}+\frac{1}{b}}{\tfrac{1}{2}\left(-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(-\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)}\right\}^{\frac{1}{2}}, \quad (11)$$

$$z=\pm\left\{\frac{-\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{\tfrac{1}{2}\left(-\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)\left(-\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)}\right\}^{\frac{1}{2}}. \quad (12)$$

Reducing these to the ordinary form, we have

$$x=\pm\left\{\frac{2abc(ab+bc-ca)}{(ab-bc+ca)(bc+ca-ab)}\right\}^{\frac{1}{2}}.$$

$$y=\pm\left\{\frac{2bca(bc+ca-ab)}{(bc-ca+ab)(ca+ab-bc)}\right\}^{\frac{1}{2}},$$

$$z=\pm\left\{\frac{2cab(ca+ab-bc)}{(ca-ab+bc)(ab+bc-ca)}\right\}^{\frac{1}{2}}.$$

Referring to equations (8), (9), and (10), of Prob. 3, Art. 120, we see that if all the letters a, b, c, x, y, and z, are reciprocated, they will agree with equations (5), (6), and (7), of this Problem.

Having discovered this relation, we could at once deduce the final expressions for x, y, and z, from those given in the Problem already referred to.

Before taking leave of this beautiful Problem, we will unite equations (5), (6), and (7), by the aid of the negative exponent. They become as follows:

$$x^{-1}y^{-1}=2^{-1}(-a^{-1}+b^{-1}+c^{-1}),$$
$$y^{-1}z^{-1}=2^{-1}(-b^{-1}+c^{-1}+a^{-1}),$$
$$z^{-1}x^{-1}=2^{-1}(-c^{-1}+a^{-1}+b^{-1}).$$

This same method of denoting a reciprocal might be used in other similar circumstances, as in equations (1), (2), (3), and in (4), &c.

We will add one more example, which will be useful to illustrate the *negative* sign.

7. A.'s age is a years, B.'s age is b years. When will A. be twice as old as B?

Let $x =$ the time sought.

At the end of x years,

A. will be $a+x$ years old.

B. will be $b+x$ years old.

Hence, we must have this condition,

$$a+x=2(b+x). \qquad (1)$$

This reduced, gives

$$x=a-2b. \qquad (2)$$

From this we see that A.'s age must, at this present time, exceed twice B.'s age, in order that x may be positive.

If A.'s age is now less than twice B.'s age, then x will be negative, and the time sought is not future, but past time.

As an illustration, suppose A. is now 30 years old, and B. is ten years old. In this case,

$$x=a-2b=30-20=10.$$

which shows that 10 years from now A. will be twice as old as B.

If A. is now 30 years old, and B. is 20 years old, then

$$x=a-2b=30-40=-10,$$

which shows that 10 years ago A. was twice as old as B. was then.

ELIMINATION BY INDETERMINATE MULTIPLIERS.

(152.) Suppose we wish to find x and y from the equations

$$2x+3y=13, \qquad (1)$$

$$5x+4y=22. \qquad (2)$$

Multiplying (1) by m, we find

$$2mx+3my=13m. \qquad (3)$$

Adding (2) and (3), we have

$$(2m+5)x+(3m+4)y=13m+22. \qquad (4)$$

This expression must be true for any value of m. If the coefficient of y, $3m+4$, be assumed equal to zero, y will disappear from (4), and we shall have $3m+4=0$, or $m=-\frac{4}{3}$. And equation (4) will give

$$x=\frac{13m+22}{2m+5}=2.$$

Again, if we had assumed $2m+5=0$, which would have given

$$m=-\tfrac{5}{2}, \qquad (5)$$

then equation (4) would have become

$$y=\frac{13m+22}{3m+4}=3. \qquad (6)$$

Now, returning to our former equations, we will subtract (2) from (3); we thus obtain

$$(2m-5)x+(3m-4)y=13m-22. \qquad (7)$$

Assume $3m-4=0$, which gives

$$m=\tfrac{4}{3}. \qquad (8)$$

This value of m causes (7) to become

$$x=\frac{13m-22}{2m-5}=2. \qquad (9)$$

Again, assume $2m-5=0$, which gives

$$m=\tfrac{5}{2}.$$

This causes (7) to become

$$y=\frac{13m-22}{3m-4}=3. \qquad (10)$$

These values of x and y are the same as just found.

It is evident that had we multiplied (2) by m and then added, or subtracted the result from (1), we should then have found, in a similar manner, the same values for x and y.

(153.) We will now apply this method to the two literal equations,

$$X_1x+Y_1y=A_1, \qquad (1)$$
$$X_2x+Y_2y=A_2. \qquad (2)$$

In these equations the capital letters are supposed to be known, and their *subscript* numerals indicate the equation to which they belong. Thus,

X_2 is the coefficient of x in the second equation.

Y_1 is the coefficient of y in the first equation.

A_2 is the absolute term, or the term independent of x and y in the second equation.

Returning to our equations, we will multiply (1) by m and add the result to (2); we thus obtain

$$(X_1m+X_2)x+(Y_1m+Y_2)y=A_1m+A_2. \qquad (3)$$

Now, assume $Y_1m+Y_2=0$, which gives

$$m=-\frac{Y_2}{Y_1}. \qquad (4)$$

This causes (3) to become

$$(X_1m+X_2)x=A_1m+A_2, \qquad (5)$$

which gives immediately

$$x=\frac{A_1m+A_2}{X_1m+X_2}=\frac{A_1Y_2-A_2Y_1}{X_1Y_2-X_2Y_1}. \qquad (6)$$

Assume $X_1m+X_2=0$, which gives

$$m=-\frac{X_2}{X_1}. \qquad (7)$$

This value of m causes (3) to become

$$y=\frac{A_1m+A_2}{Y_1m+Y_2}=\frac{A_2X_1-A_1X_2}{X_1Y_2-X_2Y_1}. \qquad (8)$$

Hence, the values of x and y are

$$\left.\begin{aligned} x&=\frac{A_1Y_2-A_2Y_1}{X_1Y_2-X_2Y_1},\\ y&=\frac{A_2X_1-A_1X_2}{X_1Y_2-X_2Y_1}.\end{aligned}\right\} \qquad (9)$$

These values of x and y may be considered as comprising the solution of all simple equations combining only two unknown quantities.

By carefully examining these general expressions, in connection with the primitive equations, we see that the values of x and y appear in a fractional form, having a common denominator, which denominator consists of the product of the coefficient of x, in the first equation, into the coefficient of y, in the second equation; diminished by the product of the coefficient of x, in the second equation, into the coefficient of y, in the first equation.

That the numerator of the expression for the value of x is derived from the common denominator, by changing the coefficients of x into the absolute terms.

The numerator of the expression for the value of y, is derived from the common denominator, by changing the coefficients of y into the absolute terms.

Hence, the solution of two simple equations may be obtained by the following

RULE.

The values of x and y will take a fractional form, having a common denominator, which common denominator may be found by multiplying the coefficient of x, in the first equation, into the coefficient of y, in the second equation; and subtracting the product of the coefficient of x, in the second equation, into the coefficient of y, in the first equation. If we conceive the absolute terms to occupy the places of the coefficients of x, and then repeat the above operation, we shall obtain the numerator for the value of x.

For the numerator of the value of y, we must conceive the absolute terms to occupy the places of the coefficients of y, and then multiply as before.

EXAMPLES.

$$1.\ \begin{cases} 3x+2y=12, \\ 4x+5y=23. \end{cases}$$

$3\times 5-4\times 2=7=$common denominator,

$12\times 5-23\times 2=14=$numerator of x,

$3\times 23-4\times 12=21=$ " " y.

Hence, $x=\frac{14}{7}=2,$
$y=\frac{21}{7}=3.$

$$2.\quad \left\{\begin{array}{l}3x-5y=5,\\4x-2y=16.\end{array}\right.$$

$3\times-2-4\times-5=14=$common denominator.
$5\times-2-16\times-5=70=$numerator of x,
$3\times16-4\times5=28=$ " " y.

Hence, $x=\frac{70}{14}=5,$
$y=\frac{28}{14}=2.$

$$3.\quad \left\{\begin{array}{l}-4x+5y=7,\\2x-3y=-5.\end{array}\right.$$

$-4\times-3-2\times5=2=$common denominator,
$7\times-3-(-5)\times5=4=$numerator of x,
$-4\times-5-2\times7=6=$ " " y.

Hence, $x=\frac{4}{2}=2,$
$y=\frac{6}{2}=3.$

$$4.\quad \left\{\begin{array}{l}-4x+5y=23,\\2x-3y=-13.\end{array}\right.$$

$-4\times-3-2\times5=2=$common denominator
$23\times-3-(-13)\times5=-4=$numerator of x,
$-4\times-13-2\times23=6=$ " " y.

Hence, $x=\frac{-4}{2}=-2,$
$y=\frac{6}{2}=3.$

5. $\left\{\begin{array}{l}4x-3y=5\\3x+4y=10\end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l}x=2,\\y=1.\end{array}\right.$

6. $\left\{\begin{array}{l}6x-5y=13\\5x+6y=21\end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l}x=3,\\y=1.\end{array}\right.$

7. $\left\{\begin{array}{l}x+3y=7\\3x-y=11\end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l}x=4,\\y=1.\end{array}\right.$

8. $\left\{\begin{array}{l} x+ay=c \\ bx+\ y=d \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=\dfrac{c-ad}{1-ab}, \\ y=\dfrac{d-bc}{1-ab}. \end{array}\right.$

9. $\left\{\begin{array}{l} x+ay=1 \\ bx+\ y=1 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=\dfrac{1-a}{1-ab}, \\ y=\dfrac{1-b}{1-ab}. \end{array}\right.$

(154.) As a still farther illustration of the method of elimination by means of indeterminate multipliers, we will proceed to the solution of three simultaneous simple equations, involving three unknown quantities, x, y, and z.

And we will continue to make use of the notation by the assistance of subscript numbers.

Let the equations be as follows:

$$X_1x+Y_1y+Z_1z=A_1, \qquad (1)$$
$$X_2x+Y_2y+Z_2z=A_2, \qquad (2)$$
$$X_3x+Y_3y+Z_3z=A_3. \qquad (3)$$

In these equations, as in those of the last example, the capital letters X, Y, Z, are the coefficients of their corresponding small letters. The small numerals placed at the base of these coefficients correspond to the particular equation to which they belong. Thus X_2 is the coefficient of x in the second equation; Y_3 is the coefficient of y in the third equation; Z_1 is the coefficient of z in the first equation, and so for the other coefficients. The letter A is used to denote the right-hand members of the equations, or the absolute terms; the subscript numbers in this case also denote the equation to which they belong.

This kind of notation, by use of subscript numbers, is very natural and simple, and combines many advantages over the ordinary methods.

Having explained this method of notation, we will now proceed to the solution of our equations.

If we multiply (1) by m, and (2) by n, and then add the results, we shall obtain

$$\left.\begin{array}{l}(X_1m+X_2n)x+(Y_1m+Y_2n)y\\ +(Z_1m+Z_2n)z=A_1m+A_2n.\end{array}\right\} \qquad (4)$$

From (4) subtracting (3), we find

$$\left.\begin{array}{l}(X_1m+X_2n-X_3)x+(Y_1m+Y_2n-Y_3)y\\ +(Z_1m+Z_2n-Z_3)z=A_1m+A_2n-A_3.\end{array}\right\} \qquad (5)$$

In order to cause y and z to vanish from this equation, we will assume

$$Y_1m+Y_2n=Y_3, \qquad (6)$$

$$Z_1m+Z_2n=Z_3. \qquad (7)$$

This assumption causes (5) to become

$$(X_1m+X_2n-X_3)x=A_1m+A_2n-A_3. \qquad (8)$$

Therefore, $$x=\frac{A_1m+A_2n-A_3}{X_1m+X_2n-X_3}. \qquad (9)$$

By Rule under Art. 153, we obtain, from (6) and (7), the values of m and n as follows:

$$m=\frac{Y_3Z_2-Y_2Z_3}{Y_1Z_2-Y_2Z_1}. \qquad (10)$$

$$n=\frac{Y_1Z_3-Y_3Z_1}{Y_1Z_2-Y_2Z_1}. \qquad (11)$$

Substituting these values of m and n in (9), we find, after a little reduction,

$$x=\frac{A_1Y_3Z_2-A_1Y_2Z_3+A_2Y_1Z_3-A_2Y_3Z_1+A_3Y_2Z_1-A_3Y_1Z_2}{X_1Y_3Z_2-X_1Y_2Z_3+X_2Y_1Z_3-X_2Y_3Z_1+X_3Y_2Z_1-X_3Y_1Z_2}. \qquad (12)$$

If we change the signs of all the terms of the numerator and denominator, and make a slight change in their arrangement, we shall have

$$x=\frac{A_1Y_2Z_3+A_2Y_3Z_1+A_3Y_1Z_2-A_1Y_3Z_2-A_2Y_1Z_3-A_3Y_2Z_1}{X_1Y_2Z_3+X_2Y_3Z_1+X_3Y_1Z_2-X_1Y_3Z_2-X_2Y_1Z_3-X_3Y_2Z_1}. \qquad (13)$$

By a similar process we shall find the values of y and z, as below:

$$y=\frac{X_1A_2Z_3+X_2A_3Z_1+X_3A_1Z_2-X_1A_3Z_2-X_2A_1Z_3-X_3A_2Z_1}{X_1Y_2Z_3+X_2Y_3Z_1+X_3Y_1Z_2-X_1Y_3Z_2-X_2Y_1Z_3-X_3Y_2Z_1} \quad (14)$$

$$z=\frac{X_1Y_2A_3+X_2Y_3A_1+X_3Y_1A_2-X_1Y_3A_2-X_2Y_1A_3-X_3Y_2A_1}{X_1Y_2Z_3+X_2Y_3Z_1+X_3Y_1Z_2-X_1Y_3Z_2-X_2Y_1Z_3-X_3Y_2Z_1}. \quad (15)$$

(155.) We will now proceed to point out some remarkable relations in the combinations of the subscript numbers, as given by equations (13), (14), and (15).

I. The denominator, which is common to the three expressions, is composed of six distinct products, each consisting of three independent factors. Three of these products are positive, and three are negative.

II. The letters forming the different products of this common denominator being always arranged in alphabetical order, X, Y, Z, we remark that the subscript numbers of the first product are 1, 2, 3. Now, if we add a unit to each of these numbers, observing that when the sum becomes 4 to substitute 1, we shall obtain 2, 3, 1, which are the subscript numbers of the second product. Again, increasing each of these by 1, observing as before to write 1 when the sum becomes 4, we find 3, 1, 2, which are the subscript numbers of the third product. If we increase each of these last numbers by 1, observing the same law, we shall obtain 1, 2, 3, which are the subscript numbers belonging to the first product. A similar method of changing has already been noticed under Art. 74.

What we have said in regard to the subscript numbers of the positive products, applies equally well in respect to the negative products.

III. The numerator of the expression for x, may be derived from the common denominator by simply substituting A for X, observing to retain the same subscript numbers.

The numerator of the expression for y may be derived from the common denominator by substituting A for Y, observing to retain the same subscript numbers.

In the same way may the numerator of the expression for z be found by changing Z of the denominator into A, retaining the same subscript numbers.

(156.) We will now proceed to show how these expressions, for x, y, and z, can be obtained by a very simple and novel process, which is easily retained in the memory, and which is applicable to all simple equations involving only three unknown quantities.

Writing the coefficients and the absolute terms in the same order as they are now placed in equations (1), (2), (3), we have

$$\left.\begin{array}{llll} X_1 & Y_1 & Z_1 & = A_1, \\ X_2 & Y_2 & Z_2 & = A_2, \\ X_3 & Y_3 & Z_3 & = A_3. \end{array}\right\} \qquad (A)$$

Now all the products of the common denominator can be found by multiplying together by threes, the coefficients which are found by passing obliquely from the left to the right, observing that if the products obtained by passing obliquely *downwards*, are taken positively, then those formed by passing obliquely *upwards* must be taken negatively, and conversely. This is in accordance with the property of the negative sign. (See Art. 17.) In the present case the products formed by passing obliquely *downwards*, are taken positive.

In this sort of checker-board movement, we must observe that when we run out at the bottom of any column, we must pass to the top of the same column; and when we run out at the top, we must pass to the bottom of the same column.

This method is most readily performed upon the blackboard, by drawing oblique lines connecting the successive factors of the different products.

We will trace out this sort of oblique movement.

Commencing with X_1, we pass obliquely downwards to Y_2, and thence to Z_3, and thus obtain the positive product of $X_1Y_2Z_3$.

Commencing with X_2, we pass obliquely downwards to Y_3; and since we have now run out with the column of Z's at the bottom, we pass to Z_1, at the top of the column, and thus obtain the positive product $X_2Y_3Z_1$.

Again, commencing with X_3, we pass to Y_1, and thence obliquely downwards to Z_2, and find the positive product $X_3Y_1Z_2$.

Now, for the negative products we make similar movements obliquely upwards.

Thus, commencing with X_1, we pass to Y_3, and thence obliquely upwards to Z_2, and find the negative product $X_1Y_3Z_2$.

Commencing with X_2, we pass obliquely upwards to Y_1, and thence to Z_3, and find the negative product $X_2Y_1Z_3$.

Again, commencing with X_3, we pass obliquely upwards to Y_2, and thence to Z_1, and thus obtain the negative product $X_3Y_2Z_1$.

Having thus obtained the denominator which is common to the values of x, y, z, we may find the numerator of the value of x, by supposing the A's to take the place of the X's, and then to repeat our checker-board movement. By changing the Y's into the A's, we shall find the numerator of the value of y; and by changing the Z's into A's, we shall find the numerator of the value of z.

(157.) We will now illustrate this method of solving simple equations containing only three unknown quantities, by a few examples.

1. Given $\left\{\begin{array}{l} 2x+3y+4z=16, \\ 3x+5y+7z=26, \\ 4x+2y+3z=19, \end{array}\right\}$ to find x, y, and z.

We will first find the common denominator.

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$2\times5\times3=30$	$2\times2\times7=-28$
$3\times2\times4=24$	$3\times3\times3=-27$
$4\times3\times7=84$	$4\times5\times4=-80$
138	-135
-135	

$3=$common denominator.

We have for the numerator of x the following operation:

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$16\times5\times3=240$	$16\times2\times7=-224$
$26\times2\times4=208$	$26\times3\times3=-234$
$19\times3\times7=399$	$19\times5\times4=-380$
847	-838
-838	

$9=$numerator, for x.

To find the numerator for y, we have

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$2\times26\times3=156$	$2\times19\times7=-266$
$3\times19\times4=228$	$3\times16\times3=-144$
$4\times16\times7=448$	$4\times26\times4=-416$
832	-826
-826	

$6=$numerator, for y.

To find the numerator for z we have

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$2\times5\times19=190$	$2\times2\times26=-104$
$3\times2\times16=96$	$3\times3\times19=-171$
$4\times3\times26=312$	$4\times5\times16=-320$
598	-595
-595	

$3=$numerator, for z.

Hence,
$$x=\tfrac{9}{3}=3.$$
$$y=\tfrac{6}{3}=2.$$
$$z=\tfrac{3}{3}=1.$$

When some of the coefficients are negative, we must observe the rule for the multiplication of signs.

2. Given $\left.\begin{array}{l}2x+4y-3z=22,\\ 4x-2y+5z=18,\\ 6x+7y-z=63,\end{array}\right\}$ to find x, y, and z.

To find the common denominator, we have

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$2\times-2\times-1=4$	$2\times7\times5=-70$
$4\times7\times-3=-84$	$4\times4\times-1=16$
$6\times4\times5=120$	$6\times-2\times-3=-36$
40	-90
-90	

-50=common denominator.

For the numerator of x, we have

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$22\times-2\times-1=44$	$22\times7\times5=-770$
$18\times7\times-3=-378$	$18\times4\times-1=72$
$63\times4\times5=1260$	$63\times-2\times-3=-378$
926	-1076
-1076	

-150=numerator, for x

Hence $x=\dfrac{-150}{-50}=3.$

Proceeding in a similar way we find the values of y and z.

3. Given $\left.\begin{array}{l}x+\frac{1}{2}y=a,\\ y+\frac{1}{3}z=a,\\ z+\frac{1}{4}x=a,\end{array}\right\}$ to find x, y, and z.

We will arrange the coefficients, omitting the unknown quantities, observing also to write 0 for such terms as are wanting.

This arrangement being made, we have

$$\begin{array}{cccc}
1 & \frac{1}{2} & 0 & = a \\
0 & 1 & \frac{1}{3} & = a \\
\frac{1}{4} & 0 & 1 & = a
\end{array}$$

$$\begin{array}{lll}
\text{POSITIVE PRODUCTS.} & & \text{NEGATIVE PRODUCTS.} \\
1\times1\times1=\ 1 & & 1\times0\times\frac{1}{3}=0 \\
0\times0\times0=\ 0 & & 0\times\frac{1}{2}\times1=0 \\
\frac{1}{4}\times\frac{1}{2}\times\frac{1}{3}=\underline{\frac{1}{24}} & & \frac{1}{4}\times1\times0=\underline{0} \\
\qquad\quad\ \frac{25}{24}=\text{common denominator.} & & \qquad\qquad\ \ 0
\end{array}$$

For the numerator of x, we have

$$\begin{array}{lll}
\text{POSITIVE PRODUCTS.} & & \text{NEGATIVE PRODUCTS.} \\
a\times1\times1=\ a & & a\times0\times\frac{1}{3}=\ \ 0 \\
a\times0\times0=\ 0 & & a\times\frac{1}{2}\times1=-\frac{1}{2}a \\
a\times\frac{1}{2}\times\frac{1}{3}=\underline{\frac{1}{6}a} & & a\times1\times0=\ \ \underline{0} \\
\qquad\qquad\ \frac{7}{6}a & & \qquad\qquad\ -\frac{1}{2}a \\
\qquad\quad\ \ \underline{-\frac{1}{2}a} & & \\
\qquad\qquad\ \frac{2}{3}a=\text{numerator, for } x. & &
\end{array}$$

Hence, $$x=\frac{2}{3}a\div\frac{25}{24}=\frac{16}{25}a.$$

By a similar process is the value of y and z found.

4. Given $\left\{\begin{array}{l} x+a(y+z)=m, \\ y+b(x+z)=n, \\ z+c(x+y)=p, \end{array}\right\}$ to find x, y, and z.

These coefficients, being properly arranged, give

$$\begin{array}{ccccc}
1 & a & a & = & m \\
b & 1 & b & = & n \\
c & c & 1 & = & p
\end{array}$$

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$1\times1\times1=\ 1$	$1\times c\times b=-bc$
$b\times c\times a=abc$	$b\times a\times1=-ab$
$c\times a\times b=abc$	$c\times1\times a=-ac$
$1+2abc$	$-ab-ac-bc$
$-ab-ac-bc$	

$1+2abc-ab-ac-bc=$common denominator.

For the numerator of x, we have

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$m\times1\times1=\ m$	$m\times c\times b=-bcm$
$n\times c\times a=acn$	$n\times a\times1=-\ an$
$p\times a\times b=abp$	$p\times1\times a=-\ ap$
$m+acn+abp$	$-bcm-an-ap$
$-bcm-an-ap$	

$m+acn+abp-bcm-an-ap=$numerator of x.

Hence, $$x=\frac{m+acn+abp-bcm-an-ap}{1+2abc-ab-ac-bc}.$$

In a similar way may the values of y and z be found. Or, the values of y and z may be at once drawn from this value of x by permutation, since the primitive equations are symmetrical, Art. 73.

This solution is much shorter than by the ordinary methods of elimination.

5. A., B., and C. owe together (a) $2190, and none of them can alone pay this sum; but when they unite, it can be done in the following ways: first, by B.'s putting $\frac{3}{7}$ of his property to all of A.'s; secondly, by C.'s putting $\frac{5}{9}$ of his property to all of B.'s; or by A.'s putting $\frac{2}{3}$ of his property to all of C.'s. How much was each worth?

Let x, y, and z represent what A., B., and C. were respectively worth.

Then we shall have these conditions:

$$x+\tfrac{3}{7}y=a,$$
$$y+\tfrac{5}{9}z=a,$$
$$z+\tfrac{2}{3}x=a.$$

Clearing these of fractions, and arranging the coefficients, we have

$$\begin{array}{cccccc} 7 & 3 & 0 & = & 7a \\ 0 & 9 & 5 & = & 9a \\ 2 & 0 & 3 & = & 3a \end{array}$$

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$7\times9\times3=189$	$7\times0\times5=0$
$0\times0\times0=\ \ 0$	$0\times3\times3=0$
$2\times3\times5=\ 30$	$2\times9\times0=0$
$219=$ common denominator.	0

For the numerator of x, we have

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$7a\times9\times3=189a$	$7a\times0\times5=\ \ 0$
$9a\times0\times0=\ \ 0$	$9a\times3\times3=-81a$
$3a\times3\times5=\ 45a$	$3a\times9\times0=\ \ 0$
$234a$	$-81a$
$-81a$	
$153a=$ numerator of x.	

$$\text{Hence,}\quad x=\frac{153a}{219}=\frac{153\times2190}{219}=1530.$$

For the numerator of y, we find

$$\begin{array}{rcr}
\text{POSITIVE PRODUCTS.} & \qquad & \text{NEGATIVE PRODUCTS.} \\
7\times 9a\times 3=189a & & 7\times 3a\times 5=-105a \\
0\times 3a\times 0=\quad 0 & & 0\times 7a\times 3=\quad 0 \\
2\times 7a\times 5=\underline{\;70a} & & 2\times 9a\times 0=\underline{\quad 0} \\
259a & & -105a \\
\underline{-105a} & & \\
154a=\text{numerator of } y. & &
\end{array}$$

Hence, $y=\dfrac{154a}{219}=1540.$

For the numerator of z, we have

$$\begin{array}{rcr}
\text{POSITIVE PRODUCTS.} & \qquad & \text{NEGATIVE PRODUCTS} \\
7\times 9\times 3a=189a & & 7\times 0\times 9a=\quad 0 \\
0\times 0\times 7a=\quad 0 & & 0\times 3\times 3a=\quad 0 \\
2\times 3\times 9a=\underline{\;54a} & & 2\times 9\times 7a=\underline{-126a} \\
243a & & -126a \\
\underline{-126a} & & \\
117a=\text{numerator of } z. & &
\end{array}$$

Hence $z=\dfrac{117a}{219}=1170.$

Collecting the results, we find that

A. was worth \$1530.
B. " \$1540.
C. " \$1170.

The student will find, after a little practice in this method, that it is much more simple than would at first sight seem.

Whenever some of the coefficients are zeros, as in the 3d and 5th examples, the work is much abridged, as in this case some of the products must become zero.

CHAPTER VIII.

MISCELLANEOUS SUBJECTS.

PROPERTIES OF NUMBERS.

(158.) Suppose the number N to be of the following form:

$$N = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \ldots\ldots + a_n r^n. \qquad (A)$$

If in this formula we suppose $r = 10$, and each of the numbers $a_0, a_1, a_2, a_3, \ldots\ldots a_n$ to be less than $r = 10$, then the above number will be expressed according to our decimal system of notation.

Thus the above formula, when adapted to the number 37854, gives

$$37854 = 4 + 5.10 + 8.10^2 + 7.10^3 + 3.10^4.$$

In this way all whole numbers may be represented by formula (A), where r denotes the base of the particular system of notation, and $a_0, a_1, a_2, a_3, \ldots\ldots a_n$ are the respective digits, counting from the right hand towards the left.

(159.) We will make use of the symbol S_d to denote the sum of all the digits; we will also represent the sum of the digits occupying the odd places, counting from the right hand towards the left, by S_{odd}, and the sum of the even digits by S_{even}.

Our formula (A) is readily changed into

$$N = S_d + a_1(r-1) + a_2(r^2-1) + a_3(r^3-1) \ldots . a_n(r^n-1).$$

Now each of the expressions $r-1$, r^2-1, r^3-1, and r^n-1, is divisible by $r-1$. Hence, if the sum of the digits is divisible by $r-1$, then will the number be divisible by $r-1$; and whatever remainder is found by dividing S_d, the same remainder will be found by dividing N.

If we take the number 3458, expressed in the decimal scale, we find $S_d=20$, which divided by 9 will give 2 remainder, and this is the same remainder as found by dividing 3458 by 9.

If we take the number 3456, we find that $S_d=18$ is divisible by 9, and consequently 3456 is also divisible by 9. The same is true of the 24 numbers found by permuting these four digits; thus,

3456 ; 3465 ; 3546 ; 3564 ; 3645 ; 3654.
4356 ; 4365 ; 4536 ; 4563 ; 4635 ; 4653.
5346 ; 5364 ; 5436 ; 5463 ; 5634 ; 5643.
6345 ; 6354 ; 6435 ; 6453 ; 6534 ; 6543.

(160.) *Hence, in general, whatever remainder is found by dividing the sum of the digits of any number by* 9, *the same remainder will be found by dividing the number itself by* 9.

(161.) *If two numbers are composed of the same digits taken in a different order, their difference will always be divisible by* 9.

For the divisibility of any number by 9, we have seen, depends upon the divisibility of the sum of its digits. Hence the difference of two numbers composed of the same digits must of necessity be divisible by 9.

(162.) What has been said in reference to the digit 9, is applicable to the digit 3, since 3 is a divisor of 9.

(163.) Fotmula (A) can readily be made to assume this form :

$$N=S_{odd}-S_{even}+a_1(r+1)+a_2(r^2-1)+a_3(r^3+1)+, \text{ \&c.}$$

Now each of the expressions $r+1$, r^2-1, r^3+1, &c., is divisible by $r+1$.

Hence, in the decimal notation, we know that if $S_{odd}-S_{even}$ is divisible by 11, then will the number be divisible by 11.

Thus, in the number 968341, we have $S_{odd}=10$, $S_{even}=21$; consequently $S_{odd}-S_{even}=-11$. Therefore the number 968341 is divisible by 11.

(164.) From the generality of formula (A), we see that there are analogous properties for each system of notation.

(165.) Suppose we wish to transform any given number expressed in the common arithmetical scale, into another number of the same value, having for the radix r, and having a_0, a_1, a_2, a_3, &c., for digits; that is, let it be required to put N into the particular form as expressed by formula (A).

If we divide each member of (A) by r, we shall obtain the remainder a_0; that is, N divided by r will give the remainder a_0, dividing again by r, and we get the remainder a_1; again, dividing by r, we find a_2 for remainder, and so on. Hence we have this

RULE.

Divide the number by the base of the new scale, and write the remainder as the unit's digit sought; divide the quotient by the base again, and write the remainder as the digit next the unit's place; proceed in this way till a quotient is obtained less than the base: this quotient is the digit of the highest order in the number in its new form. Whenever there is no remainder, 0 *is the corresponding digit.*

EXAMPLES.

1. Represent the number 2931 in the scale whose base is 8.

Following the rule, we have this

OPERATION.

$$\begin{array}{r|rl}
8 & 2931 & \\ \hline
8 & 366 & \text{- - - - } 3 = \text{first remainder} = a_0, \\ \hline
8 & 45 & \text{- - - - } 6 = \text{second remainder} = a_1, \\ \hline
 & a_3 = 5 & \text{- - - - } 5 = \text{third remainder} = a_2.
\end{array}$$

Hence, $2931 = 5.8^3 + 5.8^2 + 6.8 + 3$.

Now, if it be understood that the digits increase in an eight-fold ratio from the right hand towards the left, it may be written 5563.

2. Convert 3714 into an equivalent number having 4 for the base.

OPERATION.

$$\begin{array}{r|r l}
4 & 3714 & \\ \hline
4 & 928 & \text{- - - - } 2 = \text{first remainder} = a_0, \\ \hline
4 & 232 & \text{- - - - } 0 = \text{second remainder} = a\ , \\ \hline
4 & 58 & \text{- - - - } 0 = \text{third remainder} = a_2, \\ \hline
4 & 14 & \text{- - - - } 2 = \text{fourth remainder} = a_3, \\ \hline
 & a_5 = 3 & \text{- - - - } 2 = \text{fifth remainder} = a_4.
\end{array}$$

Hence, if we consider the digits as increasing in a four-fold ratio, our number will become

322002.

(166.) To reduce a number from any other scale into the decimal, we have this

RULE.

Multiply the digit farthest to the left by the base of the scale in which the number is expressed, and add the next digit to the product; multiply the sum again by the base, and add the third digit; proceed in this way till the unit's digit is added: the result is the number in the decimal scale. This rule is evidently the converse of the last rule.

EXAMPLE.

Let 3465 be a number expressed in the scale whose base is 9: it is required to express it in the decimal scale.

OPERATION.

$$\begin{array}{l}
3465 \\
\underline{9} \\
31 \quad = 3 \times 9 + 4, \\
\underline{9} \\
285 \quad = 31 \times 9 + 6, \\
\underline{9} \\
2570 = 285 \times 9 + 5.
\end{array}$$

Hence, 2570 is the number sought.

CONVERSION OF REPETENDS INTO VULGAR FRACTIONS.

(167.) We will commence with *simple repetends.*

And we will denote the successive figures which constitute the repetend by a_1, a_2, a_3, ... a_n.

The general form of a repetend will be

$$0.a_1a_2a_3 \ldots . a_na_1a_2a_3 \ldots . a_n, \text{ \&c.}$$

If we denote this value by s, we shall have

$$s = 0.a_1a_2a_3 \ldots . a_na_1a_2a_3 \ldots . a_n, \text{ \&c.} \qquad (1)$$

In this case we have supposed the number of figures in the repetend to be n.

If we multiply (1) by 10^n, we shall have

$$s10^n = a_1a_2a_3 \ldots . a_n.a_1a_2a_3 \ldots . a_n, \text{ \&c.} \qquad (2)$$

The right-hand member of (1) was multiplied by 10^n by removing the decimal point n places towards the right; we thus cause the decimal figures in (2) to occur after the *point* in precisely the same order as they occurred in (1).

Subtracting (1) from (2), observing that the decimal parts will cancel each other, we have

$$(10^n-1)s=a_1a_2a_3\ldots.a_n. \qquad (3)$$

Consequently, $$s=\frac{a_1a_2a_3\ldots.a_n}{10^n-1}. \qquad (A)$$

The denominator of formula (A) will consist of a succession of 9's, as many in number as there are figures in the repetend.

EXAMPLES.

1. What is the value of 0.3333, &c.? *Ans.* $=\frac{3}{9}=\frac{1}{3}$.
2. What is the value of 0.123123123, &c. $s=\frac{123}{999}=\frac{41}{333}$.
3. What is the value of 0.142857142857, &c.? *Ans.* $s=\frac{142857}{999999}=\frac{1}{7}$.

(168.) When the repetend is compound.

Let a = the repeating period.
n = the number of figures in a period.
b = the non-repeating number.
m = the number of decimal figures in b.
s = the value of the compound repetend.

Also, denote the successive figures which constitute the non-repeating part by b_1, b_2, $b_3\ldots.b_m$.

Then the general form of our *compound repetend* will be

$$s=0.b_1b_2b_3\ldots.b_ma_1a_2a_3\ldots.a_na_1a_2a_3\ldots.a_n, \text{ \&c.} \qquad (1)$$

The value of the non-repeating part is evidently

$$\frac{b_1b_2b_3\ldots.b_m}{10^m}. \qquad (2)$$

Did the part which repeats commence immediately after the decimal point, then its value could be found at once by formula

(*A*), Art. 167. But there are m decimals before the repetend commences, consequently its value is $\frac{1}{10^m}$ part of the value obtained by formula (*A*). Hence for the repeating part we find

$$\frac{a_1a_2a_3\ldots\ldots a_n}{10^m(10^n-1)}.$$

Hence, the entire value of (1) is

$$s=\frac{b_1b_2b_3\ldots\ldots b_m}{10^m}+\frac{a_1a_2a_3\ldots\ldots a_n}{10^m(10^n-1)};$$

or which is the same thing,

$$s=\frac{b(10^n-1)+a}{10^m(10^n-1)}. \qquad (B)$$

EXAMPLES.

1. What is the value of $0.017\dot{8}57142$?

In this example $n=6$; $m=3$; $a=857142$; $b=17$. These values, substituted in formula (*B*), give

$$s=\tfrac{17857125}{999999000}=\tfrac{1}{56}.$$

2. What is the value of $0.041\dot{6}$.

Ans. $s=\frac{41\times9+6}{9000}=\tfrac{375}{9000}=\tfrac{1}{24}.$

INEQUALITIES.

(169.) When two quantities or expressions are equal, as the two members of an equation, this equality is denoted by the symbol $=$.

In a similar way the symbol $>$, placed between two quantities or expressions, indicates that they are unequal, and the symbol is so placed as to open towards the greater quantity. Thus, $m>n$ shows that m is greater than n; $a<b$ shows that a is less than b.

In the theory of inequalities, *negative* quantities are considered as *less* than zero. And of two negative quantities, the one which has the greatest numerical value is considered the less. Thus,

$$0 > -4, \text{ and } -5 > -8.$$

Nearly all the principles of equations will hold good in respect to inequalities, by considering the quantities separated by the symbol $>$, as the two members of the inequality.

An inequality is said to continue in the *same sense*, when that member which was greater previous to a particular operation, remains so after the operation has been performed. And two inequalities are said to exist in the *same sense in regard to each other*, when the larger members both correspond with the left-hand members, or with the right-hand members. Thus, $a > b$ and $m > n$ exist in the same sense; so also do $x < y$ and $r < p$. But $a > b$ and $x < z$ are inequalities existing in opposite senses.

I. The same quantity may be added or subtracted from both members of an inequality, and the inequality will remain in the same sense as before.

Thus, if $6 > 2$, and we add 3 to both members, we shall have $6+3 > 2+3$; that is, $9 > 5$. Again, if $-3 > -5$, and to both members we add 2, we shall have $-3+2 > -5+2$; or, what is the same thing, $-1 > -3$; again, adding 1 to both members of this last inequality, we have $0 > -2$.

Hence, we may transpose any term of an inequality from one member to another, observing to change its *sign*.

Thus, suppose we have

$$3x+18 > 38-x,$$

we have by transposition,

$$3x+x > 38-18,$$

or

$$4x > 20.$$

II. The corresponding members of two or more inequalities existing in the same sense in respect to each other, may be added, and the resulting inequality will exist in the same sense as the given inequalities.

III. But if the corresponding members of two inequalities, existing in the same sense, be subtracted from each other, the resulting inequality will not always exist in the same sense as the given inequality.

In the two inequalities,

$$16>15, \qquad (1)$$
$$14>3, \qquad (2)$$

if we subtract the second from the first, we find

$$2<12. \qquad (3)$$

This resulting inequality exists in a contrary sense to the first and second.

IV. We can multiply or divide each member of an inequality by any positive quantity, without changing the sense of the inequality.

V. But if each member be multiplied or divided by a negative quantity, the resulting inequality will take an opposite sense.

VI. If both members of an inequality are positive, they may be raised to the same power without changing the sense of the inequality.

VII. But, when both members are not positive, they may, when raised to the same power, have the sense of their inequality changed.

(170.) We will give a few examples involving the principles of inequalities.

EXAMPLES.

1. Find the limit of the value of x in the inequality

$$4x-2>10. \qquad \textit{Ans. } x>3.$$

2. Find the limit of x in the inequality

$$3x+2<12+x. \qquad \textit{Ans. } x<5.$$

3. Given $\left\{\begin{array}{l}\dfrac{ax}{m}+bx-ab>\dfrac{a^2}{m}, \\ \dfrac{bx}{n}-ax+ab<\dfrac{b^2}{n},\end{array}\right\}$ to find the value of x.

Ans. $\Big\{$ $x>a$, and $x<b$; that is, any value between a and b will satisfy the conditions.

4. A boy being asked how many apples he had, replied, "I have more than three score, and half my number diminished by 13 is less than a score." How many had he?

These conditions expressed in symbols, are

$$x>60,$$

$$\frac{x}{2}-13<20.$$

This second condition gives

$$x<66.$$

Hence, any number between 60 and 66, exclusive, will satisfy the above conditions.

PERMUTATIONS.

(171.) The different orders in which quantities may be arranged, are called their permutations.

In our investigations we shall use the same letter, with different *subscript* numbers, to denote the quantities to be permuted. Thus, let the n quantities to be permuted, be represented by $a_1, a_2, a_3, a_4, a_5, \ldots . a_n$.

(172.) We will now proceed to determine the expressions for the number of *permutations* of n different things, when taken one and one, two and two, three and three, ...r and r together, where r is any positive integer not greater than n.

The number of permutations of n things taken separately, or one by one, is evidently equal to the number of things, or to n.

The number of permutations of n things, taken two and two together, is $n(n-1)$.

For a_1 may be placed successively before a_2, a_3, a_4, a_5, ... a_n, and thus form $(n-1)$ permutations taken two and two together; a_2 may be placed successively before a_1, a_3, a_4, a_5, ... a_n, and thus form $(n-1)$ permutations different from the former; and the same may be done with each of the quantities a_3, a_4, a_5, ... a_n, and we shall thus obtain $(n-1)$ permutations two and two together, repeated as many times as there are individual things, or n times. Hence, the total number of permutations of n things taken two and two is $n(n-1)$.

The number of permutations of n things taken three and three together, is equal to $n(n-1)(n-2)$.

For we have just shown that the number of permutations of n things two by two is $n(n-1)$, therefore the number of permutations of a_2, a_3, a_4, a_5 .... a_n, taken two and two, is found by diminishing n by a unit in the expression $n(n-1)$, which thus becomes $(n-1)(n-2)$. Now, writing a_1 before each of these permutations, we shall find $(n-1)(n-2)$ permutations of three by three, all commencing with a_1. It is obvious the same thing may be done with all the other letters, a_2, a_3, a_4, a_5 .... a_n. Therefore, the total number of permutations of n things taken three by three is $n(n-1)(n-2)$.

By a similar process we should be able to show that the number of permutations of n things taken four and four is

$$n(n-1)(n-2)(n-3).$$

(173.) ***By this method of induction, we infer that the number of permutations of n things taken r and r together, will be given by the continued product of the natural numbers descending from n, until as many factors are used as there are things in each permutation.***

The general expression will be

$$n(n-1)(n-2)(n-3)\ldots\ldots(n-r+1). \qquad (A)$$

If we suppose $r=n$, then all the different things will be found in each permutation, and formula (A) will become, where the order of its factors is reversed,

$$1\times2\times3\times4\times5\ldots\ldots(n-1)n, \qquad (B)$$

which is the product of all the natural numbers as far as n.

EXAMPLES.

1. How many permutations can be made of six individual things, all being taken at a time?

Ans. $1\times2\times3\times4\times5\times6=720$.

2. In how many different ways may 15 persons sit at table?

Ans. $1\times2\times3\times4\times5\times6\times7\times8\times9\times10\times11\times12\times13\times14\times15=1307674368000$.

3. How many changes can be rung on 8 bells?

Ans. $1\times2\times3\times4\times5\times6\times7\times8=40320$.

(174.) Let us endeavor to find the number of permutations of n things, when any assigned number of them become identical.

Thus let us find an expression for the number of permutations of n things, r of which are identical with each other.

We have already found the number of permutations of n things all taken together, supposing them all different from each other, to be

$$n(n-1)\ldots\ldots3.2.1$$

If r of these quantities become identical, the permutations which arise by their interchange with each other, or from their *particular* permutations, which are $1\times2\times3\ldots.r$ in number, *for any assigned position of the other letters*, are reduced to one. The number of permutations, therefore, when all the letters are different from each other, is $1\times2\times3\ldots.r$ times as great as when r of them become identical. Or, in other words,

$$\frac{n(n-1)\times\ldots.2\times1}{1\times2\times\ldots.r},$$

is the expression for the number of permutations under the circumstances supposed.

If, in addition to r quantities, which become identical, there are s others, which, though different from the former, are still identical with each other, then there are $1\times2\times3\times\ldots.s$ permutations corresponding to their interchange with each other, which are reduced to one, for any given position of the other quantities. The expression for the number of permutations, under these circumstances, becomes

$$\frac{n(n-1)\times\ldots.2\times1}{1\times2\times\ldots.r\times1\times2\times\ldots.s}.$$

The same reasoning obviously applies to any number of classes of letters or things which become identical with each other; and consequently if, of n quantities, r_1 are of one kind, r_2 of another, r_3 of a third, and so on, as far as r_m of the mth class, then their whole number of permutations will be expressed by

$$\frac{n(n-1)(n-2)\times\ldots.3\times2\times1}{1\times2\ldots.r_1\times1\times2\ldots.r_2\times1\times2\ldots.r_3\times\ldots.1\times2\ldots.r_m}.$$

(175.) We will illustrate this formula by a few

EXAMPLES.

1. Find the number of permutations (p) of the letters in the word *Algebra.*

In this case $n=7$, and the letter a appears twice; consequently,

$$p=\frac{7.6.5.4.3.2.1}{1.2}=2520.$$

2. Find the number of permutations (p) of the letters in the word *perseverance.*

In this case $n=12$, and the letter e appears four times, and r twice. Therefore,

$$p=\frac{12.11.10.9.8.7.6.5.4.3.2.1}{1.2.3.4\times1.2}=9979200.$$

3. In how many different ways may three maple-trees, five ash-trees, and two elm-trees, be set out in a single row?

Ans. 2520.

COMBINATIONS.

(176.) By *combinations* of different letters or quantities, we mean the different collections which can be made of any assigned number of them, without reference to the order of their arrangement. Thus, a_1a_2, a_1a_3, and a_2a_3, are the only different combinations of the three letters a_1, a_2, a_3, taken two and two together, which form *six* different permutations. There is only *one* combination of the same three letters, taken all together, though they form *six* different permutations.

(177.) We will now proceed to determine the number of combinations of n things, taken r and r together, when r is less than n.

The number of combinations of n things, taken separately or one and one together, is clearly n.

The number of combinations of n things, taken two and two together, is $\frac{n(n-1)}{1.2}$.

For the number of permutations of n things taken two and two together, is $n(n-1)$, Art. 172; and there are two permutations (a_1a_2, a_2a_1) corresponding to one combination; the number of combinations will be found, therefore, by dividing the number of permutations by 2, or 1.2.

The number of combinations of n things taken three and three together, is

$$\frac{n(n-1)(n-2)}{1.2.3}.$$

For the number of permutations of n things, taken three and three together, is $n(n-1)(n-2)$, Art. 172, and there are 1.2.3 permutations for one combination of three things. The number of combinations will therefore be found by dividing the number of permutations by 1.2.3.

By the same reasoning we might show that the number of combinations of n things taken r and r together, is

$$\frac{n(n-1)(n-2)\ldots.(n-r+1)}{1.2.3\ldots.r}. \qquad (D)$$

(178.) We will illustrate this formula by a few simple

EXAMPLES.

1. How many lottery tickets, each having three numbers, can be formed out of 60 numbers? *Ans.* $\frac{60\times59\times58}{1\times2\times3}=34220.$

2. How many diagonals can be drawn in a polygon of n sides?

It is evident that the whole number of lines which can be drawn by joining the corners of the polygon is equal to the number of combinations of n things, taken two and two, since a line can be drawn from any point to any other point.

Hence, the whole number of lines thus drawn is

$$\frac{n(n-1)}{1\,.\,2}.$$

But n of these lines must be required to form the sides of the polygon. Therefore the number of diagonals is

$$\frac{n(n-1)}{1\,.\,2}-n=\frac{n(n-3)}{2}.$$

3. How many different triangles can be formed by joining the corners of a polygon of n sides, counting only such triangles as have their vertices corresponding with the corners of the polygon?

Now, as three points will determine the position of a triangle, it is evident there must be as many triangles as the number of combinations of n things taken three and three together, which is

$$\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}.$$

PROBABILITIES.

(179.) There is scarcely any subject which admits of a greater variety of instructive illustrations than the *Theory of Combinations*. For it is by the aid of this theory that many nice calculations are made in reference to chances and probabilities. We will for this reason make a few remarks, in this place, concerning the method of applying algebra to the calculation of probabilities.

(180.) If a expresses the number of favorable events or cases, and b the number of those which are unfavorable, the *chance* of the favorable event or of the required case existing, is expressed by

$$\frac{a}{a+b},$$

while the *chance* of an unfavorable event or of the required case not existing, is expressed by

$$\frac{b}{a+b}.$$

(181.) From this mode of representation, it will follow that *certainty*, which supposes all the events favorable, in the first case, when $b=0$; or all of them unfavorable, in the second case, when $a=0$, will be expressed by 1. The ratio, therefore, of the *chances* to *certainty*, or of the degree of probability to certainty, will be the ratio which the fraction, by which it is denoted, bears to unity, or the ratio of its numerator to its denominator.

(182.) The ratio of the chances of success to that of failure, or the ratio of the *odds for or against*, as expressed in popular language, will be that of a to b, or of b to a, which are the numerators of the fractions by which the respective chances are denoted.

(183.) The following are examples of the representation of simple chances:

1. What is the chance of throwing an ace with a single die?

There is only one face which can be uppermost; that is, there is but one *favorable* case, while there are five *unfavorable* cases. The *chance*, therefore, that this face is the *ace*, is $\frac{1}{6}$. The chance that this face is not the *ace* is $\frac{5}{6}$, for there are five out of six equally possible ways, which are favorable to this last case.

The chance that the face thrown is either an *ace* or a *deuce*, is $\frac{2}{6}$ or $\frac{1}{3}$. For there are here two favorable cases out of six which are equally likely to happen. The chance that it is neither an *ace* nor a *deuce* is $\frac{4}{6}=\frac{2}{3}$.

If the die had been a regular tetrahedron, whose faces were marked with the numbers 1, 2, 3, 4, the chance of its *resting upon an ace* would be $\frac{1}{4}$, the chance of its not doing so would be $\frac{3}{4}$.

The chance of drawing the ace of spades from a pack of 52 cards is $\frac{1}{52}$. The chance of drawing any one of the four aces is $\frac{4}{52}=\frac{1}{13}$; for there are four favorable cases out of fifty-two which are both favorable and unfavorable, and all of them are equally likely to happen.

(184.) If 14 white and 6 black balls be thrown into an urn, the chance of drawing a white ball out of it, at one trial, is $\frac{14}{20}=\frac{7}{10}$. The chance of failure, or of drawing a black ball, is $\frac{6}{20}=\frac{3}{10}$.

In the preceding, and in all other cases, the chances of success and failure of the *same* event are supplemental to each other, their sum being equal to 1, which is the measure of *certainty*. The knowledge of one, therefore, necessarily determines the other.

(185.) The following are some examples of *compound chances:*

To find the chance of throwing an ace twice in succession with a single die.

There are six cases which are equally likely to occur at the first throw, and the same number at the second. These may be combined or permuted together in $6\times6=36$ different ways which are equally likely to happen, and only one of them is *favorable*. The chance is therefore $\frac{1}{36}$.

The chance of throwing *two aces* at one *contemporaneous* throw with two dice, is equally $\frac{1}{36}$; for the succession of time makes no difference whatever in the number of favorable and unfavorable permutations.

The chance of throwing an *ace* at the first throw, and a *deuce* at the second, is also $\frac{1}{36}$; for there is only one favorable permutation out of 36.

The chance of throwing an *ace* at one throw, and a *deuce* at the other, *without reference to their order of succession*, is $\frac{1}{18}$: for in this case there are two permutations forming one combination, (1, 2 and 2, 1), which are favorable to the hypothesis made, and two *only* out of the whole 36.

(186.) The chance of an event contingent upon other events, is the continued product of the chances of the separate events.

Let the several chances be

$$\frac{a_1}{a_1+b_1};\ \frac{a_2}{a_2+b_2};\ \frac{a_3}{a_3+b_3};\ \ldots\ldots\frac{a_n}{a_n+b_n};$$

where a_1, a_2, a_3, ... a_n, represent the numbers of cases which are favorable, and b_1, b_2, b_3, ... b_n the numbers of cases which are unfavorable, to the particular hypothesis made in each separate event, whether of success or failure.

We will consider, in the first instance, the chance which is dependent upon the two separate chances

$$\frac{a_1}{a_1+b_1} \text{ and } \frac{a_2}{a_2+b_2}.$$

Every case in a_1+b_1 may be combined with every case in a_2+b_2, and thus form $(a_1+b_1)(a_2+b_2)$ combinations of cases, which are equally likely to happen.

The favorable cases in the first (a_1) may be combined severally with the favorable cases in the second (a_2), and thus form $a_1 \times a_2$ combinations of case favorable to the compound event.

The compound chance is therefore denoted by

$$\frac{a_1 \times a_2}{(a_1+b_1)(a_2+b_2)},$$

which is the product of the separate chances.

We will now consider the chance of the event contingent upon three other events, whose respective chances are

$$\frac{a_1}{a_1+b_1};\ \frac{a_2}{a_2+b_2};\ \frac{a_3}{a_3+b_3}.$$

The several combinations of all the cases in the two last chances, which are by the last case, $(a_1+b_1)(a_2+b_2)$ in number, may be severally combined with the a_3+b_3 different cases both favorable and unfavorable, of the third chance, and thus form $(a_1+b_1)(a_2+b_2)(a_3+b_3)$ combinations which are equally likely to happen.

The favorable cases in the two first chances, which are $a_1\times a_2$ in number, may be combined severally with the a_3 favorable cases of the third chance, and thus form $a_1\times a_2\times a_3$ cases which are favorable to the compound event.

The chance, therefore, of the compound event is

$$\frac{a_1\times a_2\times a_3}{(a_1+b_1)(a_2+b_2)(a_3+b_3)},$$

which is the product of the simple chances.

It is obvious that the chance of a compound event depending upon a greater number of simple events may be found in the same way.

This is a most important proposition in the Doctrine of Chances, and makes the calculation of the chance of any compound event dependent upon the separate and simple chances of the several events, in their assigned order, upon which it is dependent.

(187.) We will add some cases of compound chances.

1. What is the chance of throwing an ace in the first *only* of two successive throws?

The first simple chance is $\frac{1}{6}$.

The second simple chance is $\frac{5}{6}$; for an ace must *not* be thrown the second time, and there are five *favorable* cases for its failure.

The compound chance is therefore $\frac{1}{6}\times\frac{5}{6}=\frac{5}{36}$.

2. What is the chance of drawing the four aces from a pack of cards in four successive trials?

The first simple chance is $\frac{4}{52}$.

The second simple chance is $\frac{3}{51}$.

For if an ace be drawn the first time, there will remain only 3 aces and 51 cards.

The third simple chance is $\frac{2}{50}$.

For if two aces be drawn the two first times, there will remain only two aces and 50 cards.

The fourth simple chance is, by the above reasoning, $\frac{1}{49}$.

The compound chance required is, therefore,

$$\frac{4.3.2.1}{52.51.50.49}=\frac{1}{270725}.$$

THE END.

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